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Tropical geometry, the motivic nearby fiber, and limit mixed Hodge numbers of hypersurfaces

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Abstract

A complex variety, degenerating over a punctured disk, carries a limit mixed Hodge structure on its cohomology, encoding the action of monodromy. The associated limit mixed Hodge–Deligne polynomial can be expressed in terms of the motivic nearby fiber. Using techniques from tropical geometry, we present a new formula for the motivic nearby fiber and concentrate on the case of degenerating families of complex hypersurfaces, generalizing work of Danilov and Khovanskii and Batyrev and Borisov. If these families satisfy a natural smoothness condition, called schönness, their limit mixed Hodge–Deligne numbers can be expressed in terms of new, combinatorial invariants of a polyhedral subdivision of the associated Newton polytope. These invariants are multivariable, Ehrhart-theoretic extensions of Stanley’s invariants of subdivisions and are situated in his theory in a companion combinatorial paper.

Keywords: Tropical geometry, Monodromy, Hodge theory, Polytopes, Ehrhart theory, Intersection cohomology

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1 Background

Let \mathcal{O} be the ring of germs of analytic functions in \mathbb{C} in a neighborhood of the origin, and let \mathbb{K} be its field of fractions. A variety X over \mathbb{K} is naturally interpreted as a family of complex varieties $f : X \rightarrow \mathbb{D}^*$ where \mathbb{D}^* is a small punctured disk about the origin over which X is defined. After possibly shrinking \mathbb{D}^* , we may assume that $X \rightarrow \mathbb{D}^*$ is a locally trivial fibration, and we fix a nonzero fiber $X_{\text{gen}} := f^{-1}(t)$ for some $t \in \mathbb{D}^*$.

Our goal is to compute an important invariant of X called the **motivic nearby fiber** $\psi_X = \psi_f$, which was introduced by Denef and Loeser [23] and contains information about the extension of f to a family over the whole complex disk \mathbb{D} . Moreover, the motivic nearby fiber specializes to the limit Hodge–Deligne polynomial of X and to both the χ_Y -characteristic and Euler characteristic of X_{gen} .

The motivic nearby fiber is ‘additive’ in the following sense. For any field k , the **Grothendieck ring** $K_0(\text{Var}_k)$ of algebraic varieties over k is the free abelian group generated by isomorphism classes $[V]$ of varieties V over k , modulo the relation

$$[V] = [U] + [V \setminus U],$$

whenever U is an open subvariety of V . Multiplication is defined by

$$[V] \cdot [W] = [V \times W].$$

We will follow the convention that $\mathbb{L} := [\mathbb{A}^1]$. A **motivic invariant** over k is a ring homomorphism $K_0(\text{Var}_k) \rightarrow R$, for some ring R . The motivic nearby fiber is a ring homomorphism

$$\psi : K_0(\text{Var}_{\mathbb{K}}) \rightarrow K_0(\text{Var}_{\mathbb{C}}), \quad \psi([X]) = \psi_X. \quad (1)$$

We briefly recall the construction of the motivic nearby fiber and refer the reader to [14] for details. By Hironaka’s resolution of singularities [34] and induction on dimension, if k has characteristic zero, then $K_0(\text{Var}_k)$ is generated by the classes of smooth, proper varieties. If X is smooth and proper, then by [42] there exists a **semi-stable reduction** of X . That is, after possibly pulling back the family $f : X \rightarrow \mathbb{D}^*$ by a map $\mathbb{D}^* \rightarrow \mathbb{D}^*$ ramified over the puncture, there exists an extension of f defined over \mathbb{D} such that the central fiber is a reduced, simple normal crossings divisor with irreducible components $\{D_i\}_{i \in \{1, \dots, r\}}$. If for every non-empty subset $I \subseteq \{1, \dots, r\}$, we set $D_I^\circ = \cap_{i \in I} D_i \setminus \cup_{j \notin I} D_j$, then

$$\psi_X = \sum_{\emptyset \neq I \subseteq \{1, \dots, r\}} [D_I^\circ] (1 - \mathbb{L})^{|I|-1}.$$

The independence of the motivic nearby fiber under ramified base change is proved in [46, Sec. 6], and the fact that the motivic nearby fiber is well defined as a map of Grothendieck groups is shown in [14, Sec. 8]. We will give an approach to computing the motivic nearby fiber via tropical geometry.

A result of Luxton and Qu [43, Theorem 6.11] that was conjectured by Tevelev in [60] states that every variety X over \mathbb{K} contains an open, very affine subvariety X° that is **schön** in the sense of Tevelev [60, Definition 1.1]. Here X° being very affine means that it can be embedded as a closed subvariety of $(\mathbb{K}^*)^n$ for some n , defined by an ideal $I \subseteq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. In this case, X° being schön means that for every $w \in \mathbb{R}^n$, the corresponding initial degeneration $\text{in}_w X^\circ$ defined by the ideal $\text{in}_w I := (\text{in}_w(f) \mid f \in I) \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of initial degenerations is a smooth subvariety of $(\mathbb{C}^*)^n$ [33, Prop 3.9]. For a more geometric description of $\text{in}_w X^\circ$, we refer the reader to Sect. 2. The notion of

schönness of a hypersurface of a complex torus was introduced by Khovanskiĭ in [41] as a hypersurface *non-degenerate* with respect to its Newton polytope.

Luxton and Qu's result immediately implies that the Grothendieck ring $K_0(\text{Var}_{\mathbb{K}})$ is generated by schön subvarieties of tori. In particular, to describe the motivic nearby fiber, in principle, we may reduce to the case of a schön subvariety of a torus. *In what follows, we will always assume that $X^\circ \subseteq (\mathbb{K}^*)^n$ is schön.*

Definition 1.1 The tropical variety $\text{Trop}(X^\circ)$ associated with X° is the set of points

$$\{w \in \mathbb{R}^n \mid \text{in}_w X^\circ \neq \emptyset\}.$$

The tropical variety $\text{Trop}(X^\circ)$ can be given a rational polyhedral structure Σ such that the initial degeneration at $w \in \text{Trop}(X^\circ)$ only depends on the cell containing w in its relative interior (this follows from [43, Theorem 1.5]). Hence for every cell γ of Σ , we may define $[\text{in}_\gamma X^\circ] := [\text{in}_w X^\circ] \in K_0(\text{Var}_{\mathbb{C}})$ for any $w \in \mathbb{R}^n$ in the relative interior of γ . Our main result is as follows:

Theorem 1.2 *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön closed subvariety and let Σ be a rational polyhedral structure on $\text{Trop}(X^\circ)$. Then the motivic nearby fiber ψ_{X° is given by*

$$\psi_{X^\circ} = \sum_{\substack{\gamma \in \Sigma \\ \gamma \text{ bounded}}} (-1)^{\dim \gamma} [\text{in}_\gamma X^\circ].$$

We provide a proof of Theorem 1.2 in Sect. 2. The theorem immediately gives expressions for the motivic nearby fiber of various partial compactifications of X° that are not smooth in general (Corollary 2.4). In the case of smooth compactifications of X° , this result generalizes [38, Theorem 5.1] (see Theorem 2.3, Remark 2.5 below). In particular, in contrast to the above theorem, [38, Theorem 5.1] requires that Σ has a unimodular recession fan.

A key point is that there exist explicit algorithms to compute both the initial degenerations of X° and its tropical variety with a choice of rational polyhedral structure. Moreover, there is a range of available software that implements these algorithms [36, 37]. Hence given any variety over \mathbb{K} , if one is able to produce a stratification into locally closed, very affine schön subvarieties, as guaranteed by Luxton and Qu's result, then the above theorem gives a practical approach to computing the motivic nearby fiber.

Example 1.3 For a concrete example, let t be a local coordinate on \mathbb{D} , and let

$$C^\circ = \{(x, y) \in (\mathbb{K}^*)^2 \mid t(1 + x^4 + y^4) + xy(1 + x + y) = 0\}.$$

Then C_{gen}° is a genus 3 curve with 12 points removed. The corresponding tropical variety has a polyhedral structure with four vertices $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -1)$, and $v = (0, 0)$, six bounded edges joining each pair of vertices, and three unbounded edges emanating from each v_i in the direction of v_i . The initial degeneration at each v_i , at v , and at each bounded edge is isomorphic to \mathbb{A}^1 minus 6, 2, and 1 point, respectively. Theorem 1.2 then implies that

$$\psi_{C^\circ} = 3(\mathbb{L} - 6) + (\mathbb{L} - 2) - 6(\mathbb{L} - 1) = -14 - 2\mathbb{L}.$$

Observe that by composing the motivic nearby fiber map (1) with a motivic invariant over \mathbb{C} , we obtain a new motivic invariant over \mathbb{K} , to which we may apply our formula.

In particular, as described in detail in Sect. 3, if $E : K_0(\text{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}[u, v]$ denotes the Hodge–Deligne map, then we obtain a series of well-known invariants:

$$K_0(\text{Var}_{\mathbb{K}}) \xrightarrow{\psi} K_0(\text{Var}_{\mathbb{C}}) \xrightarrow{E} \mathbb{Z}[u, v] \xrightarrow{v \mapsto 1} \mathbb{Z}[u] \xrightarrow{u \mapsto 1} \mathbb{Z}. \quad (2)$$

For any variety X over \mathbb{K} , the polynomial $E(X_{\infty}; u, v) := E(\psi([X]))$ is called the **limit Hodge–Deligne polynomial** of X and encodes information on the variation of mixed Hodge structures of the family $X \rightarrow \mathbb{D}^*$. The specialization obtained by setting $v = 1$ is the χ_y -**characteristic** $E(X_{\text{gen}}; u, 1) = E(X_{\infty}; u, 1)$ of X_{gen} and encodes information about the Hodge filtration on the cohomology with compact supports of X_{gen} . Finally, the specialization $e(X_{\text{gen}}) = E(X_{\text{gen}}; 1, 1)$ is the familiar **topological Euler characteristic** of X_{gen} . Theorem 1.2 immediately provides formulas for these invariants in the case when X° is schön.

Corollary 1.4 *Let $X^{\circ} \subseteq (\mathbb{K}^*)^n$ be a schön closed subvariety and let Σ be a rational polyhedral structure on $\text{Trop}(X^{\circ})$. Let $\text{vert}(\Sigma)$ denote the set of vertices of Σ . If we fix a nonzero fiber $X_{\text{gen}}^{\circ} := f^{-1}(t)$ for some $t \in \mathbb{D}^*$, then the limit Hodge–Deligne polynomial of X° is given by*

$$E(X_{\infty}^{\circ}; u, v) = \sum_{\substack{\gamma \in \Sigma \\ \gamma \text{ bounded}}} (-1)^{\dim \gamma} E(\text{in}_{\gamma} X^{\circ}; u, v),$$

the χ_y -characteristic of X_{gen}° is given by

$$E(X_{\text{gen}}^{\circ}; u, 1) = \sum_{\substack{\gamma \in \Sigma \\ \gamma \text{ bounded}}} (-1)^{\dim \gamma} E(\text{in}_{\gamma} X^{\circ}; u, 1),$$

and the Euler characteristic $e(X_{\text{gen}}^{\circ})$ of X_{gen}° is given by

$$e(X_{\text{gen}}^{\circ}) = \sum_{\gamma \in \text{vert}(\Sigma)} e(\text{in}_{\gamma} X^{\circ}).$$

Here the last equality follows from the fact that the Euler characteristic $e(\text{in}_{\gamma} X^{\circ})$ is zero unless γ is a vertex of Σ (see (3)).

As discussed above, this corollary provides a strategy to compute any of these invariants. For example, if one wants to compute the Euler characteristic of a complex variety V and then if one can find a stratification of V into locally closed pieces, each of which can be realized as the general fiber of a schön degeneration, then the above corollary reduces the problem to finding the Euler characteristic of a set of ‘simpler’ complex varieties.

Before presenting our main application, we introduce a new motivic invariant over \mathbb{K} (see Sect. 3 for details). Given a variety X over \mathbb{K} , consider the complex cohomology with compact supports $H_c^m(X_{\text{gen}})$ of the fiber X_{gen} . Then $H_c^m(X_{\text{gen}})$ admits three natural filtrations. Firstly, since it is a complex variety, it admits a decreasing filtration F^{\bullet} called the **Hodge filtration** and an increasing filtration W_{\bullet} called the **Deligne weight filtration**. Secondly, the monodromy map $T : H_c^m(X_{\text{gen}}) \rightarrow H_c^m(X_{\text{gen}})$ can be written as $T = T_s T_u$, where T_s is semi-simple and T_u is unipotent, and we may consider the action of the nilpotent operator $N = \log T_u$ on $H_c^m(X_{\text{gen}})$. A result of Steenbrink and Zucker [59] and El Zein [24] states that $H_c^m(X_{\text{gen}})$ admits an increasing filtration M_{\bullet} called the **monodromy weight filtration**, such that the filtration induced by M_{\bullet} on the quotient $Gr_r^W H_c^m(X_{\text{gen}})$ encodes the Jordan block structure of the induced action of N on $Gr_r^W H_c^m(X_{\text{gen}})$. Note

that M_\bullet is not the filtration encoding the Jordan block structure of the induced action of N on the whole of $H_c^m(X_{\text{gen}})$, but rather some kind of convolution of this filtration with W_\bullet .

Here, we use $H_c^m(X_\infty)$ to mean compactly supported cohomology equipped with the Hodge, monodromy, (and possibly also weight) filtrations. We will refer to the corresponding invariants

$$h^{p,q,r}(H_c^m(X_\infty)) = \dim(Gr_F^p Gr_{p+q}^M Gr_r^W H_c^m(X_\infty)),$$

as the **refined limit mixed Hodge numbers**. Summing over q or r recovers the (usual) mixed Hodge numbers and the limit mixed Hodge numbers of $H_c^m(X_{\text{gen}})$, respectively (see (6) and (7)). We define a polynomial called the **refined limit Hodge–Deligne polynomial** by

$$E(X_\infty; u, v, w) = \sum_{p,q,r} \sum_m (-1)^m h^{p,q,r}(H_c^m(X_\infty)) u^p v^q w^r,$$

and show that we have an induced motivic invariant

$$\bar{E} : K_0(\text{Var}_{\mathbb{K}}) \rightarrow \mathbb{Z}[u, v, w], [X] \mapsto E(X_\infty; u, v, w).$$

We now have a commutative diagram of motivic invariants

$$\begin{array}{ccccc} K_0(\text{Var}_{\mathbb{K}}) & \xrightarrow{\bar{E}} & \mathbb{Z}[u, v, w] & \xrightarrow[u \mapsto uw^{-1}]{v \mapsto 1} & \mathbb{Z}[u, w] \\ \downarrow \psi & & \downarrow w \mapsto 1 & & \downarrow w \mapsto 1 \\ K_0(\text{Var}_{\mathbb{C}}) & \xrightarrow{E} & \mathbb{Z}[u, v] & \xrightarrow[v \mapsto 1]{} & \mathbb{Z}[u] \xrightarrow[u \mapsto 1]{} \mathbb{Z}, \end{array}$$

where the first vertical arrow together with the lower horizontal row coincides with (2), and we have corresponding invariants

$$\begin{array}{ccccccc} [X] & \longrightarrow & E(X_\infty; u, v, w) & \longrightarrow & E(X_{\text{gen}}; u, w) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \psi_X & \longrightarrow & E(X_\infty; u, v) & \longrightarrow & E(X_{\text{gen}}; u, 1) & \longrightarrow & e(X_{\text{gen}}), \end{array}$$

where $E(X_{\text{gen}}; u, w)$ is the **Hodge–Deligne polynomial** of X_{gen} . One may think that every successive specialization forgets about a filtration in the following sense: The invariants $E(X_\infty; u, v, w)$, $E(X_\infty; u, v)$, $E(X_{\text{gen}}; u, w)$ and $E(X_{\text{gen}}; u, 1)$ encode information about the filtrations $(F^\bullet, W_\bullet, M_\bullet)$, (F^\bullet, M_\bullet) , (F^\bullet, W_\bullet) and F^\bullet , respectively.

For the remainder of the introduction, we assume that $X^\circ \subseteq (\mathbb{K}^*)^n$ is a schön hypersurface. In this case, the Hodge–Deligne polynomial $E(X_{\text{gen}}^\circ; u, w)$ encodes precisely the (usual) mixed Hodge numbers of X_{gen}° , and its computation is a classical problem. Indeed, an algorithm to compute the mixed Hodge numbers of a schön hypersurface of a complex torus was given by Danilov and Khovanskii in [20]. Much later, using deep results from intersection cohomology, a combinatorial formula was given by Batyrev and Borisov and was the key technical result in their construction of mirror Calabi–Yau varieties in [8]. A cleaner combinatorial formula was later given by Borisov and Mavlyutov in [15]. Finally, a combinatorial proof of the Borisov–Mavlyutov formula was given by the second author in [56], as part of work giving a representation-theoretic generalization.

Our main application is a combinatorial formula for the refined limit mixed Hodge numbers of the schön hypersurface X° . In this case, this is equivalent to giving a combinatorial

formula for the refined limit Hodge–Deligne polynomial $E(X_\infty^\circ; u, v, w)$. In particular, by specializing, we obtain a combinatorial formula for the limit mixed Hodge numbers of X° . Our result also specializes to give the Borisov–Mavlyutov formula for the usual mixed Hodge numbers of X_{gen}° . Although we make use of the strategy of the Danilov–Khovanskii algorithm, our proof is self-contained and only relies on Theorem 1.2 together with some new combinatorics. In particular, in Sect. 5.2, using the theory of valuations of polytopes (see, e.g., [45]), we present a new proof of a formula of Danilov–Khovanskii [20, Section 4] for the χ_Y -characteristic of X_{gen}° . Since the necessary combinatorial results are involved and we expect them to be of outside interest, we will only quote them as needed and defer all proofs and discussion to [39]. We will mention that some of these results build on the work of Stanley [55], together with recent work of Athanasiadis and Savvidou [2, 3], and Nill and Schepers [47].

As explained in Sect. 5.1, we may associate with X° its corresponding Newton polytope P together with a corresponding regular, lattice polyhedral subdivision \mathcal{S} . In [39, Section 9], we introduce a combinatorial invariant $h^*(P, \mathcal{S}; u, v, w) \in \mathbb{Z}[u, v, w]$ called the **refined limit mixed h^* -polynomial** of (P, \mathcal{S}) , which only depends on the poset structure of \mathcal{S} , together with the number of lattice points in all dilates of all cells of \mathcal{S} . This invariant has several interesting specializations. In particular, $h^*(P, \mathcal{S}; u, 1, 1) = h^*(P; u)$ is the usual h^* -polynomial of P , encoding the number of lattice points in all dilates of P [10].

Theorem 1.5 *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, \mathcal{S}) and $\dim P = n$. Then the refined limit Hodge–Deligne polynomial of X° is given by*

$$uvw^2 E(X_\infty^\circ; u, v, w) = (uvw^2 - 1)^{\dim P} + (-1)^{\dim P+1} h^*(P, \mathcal{S}; u, v, w).$$

As discussed above, Theorem 1.5 immediately gives explicit combinatorial formulas for the refined limit mixed Hodge numbers and limit mixed Hodge numbers of X° (see Corollary 5.11). In particular, we deduce that these invariants only depend on the pair (P, \mathcal{S}) , and not on the specific choice of X° . Our results allow one to compute the refined limit Hodge–Deligne polynomial of various compactifications of X° but not necessarily the refined limit mixed Hodge numbers as we elaborate in Remark 5.12. In Example 5.13, we apply our results to obtain formulas for **stringy invariants** associated with families of Calabi–Yau varieties.

Example 1.6 When $n = 2$, $X^\circ \subseteq (\mathbb{K}^*)^2$ may be viewed as a family of non-compact, smooth curves. Let (P, \mathcal{S}) denote the corresponding pair consisting of a lattice polytope in a lattice M together with a lattice polyhedral subdivision. In this case, Theorem 1.5 has the following explicit description. Let ∂P and $\text{Int}(P)$ denote the boundary and interior of P , respectively. Then the coefficients of uvw^3 and uv^2w^3 in $h^*(P, \mathcal{S}; u, v, w)$ are, respectively, given by

$$\begin{aligned} h_{0,0,1}^*(P, \mathcal{S}) &= \sum_{\substack{F \in \mathcal{S}, F \not\subseteq \partial P \\ \dim F \leq 1}} \#(\text{Int}(F) \cap M), \\ h_{0,1,1}^*(P, \mathcal{S}) &= \sum_{\substack{F \in \mathcal{S} \\ \dim F = 2}} \#(\text{Int}(F) \cap M), \end{aligned}$$

and one can compute

$$E(X_\infty^\circ; u, v, w) = 1 - \#(\partial P \cap M) - h_{0,0,1}^*(P, \mathcal{S})(1 + uv)w - h_{0,1,1}^*(P, \mathcal{S})(u + v)w + uvw^2.$$

If X denotes the closure of X° in the toric variety over \mathbb{K} corresponding to the normal fan of P , then X may be viewed as a family of smooth, compact curves with

$$E(X_\infty; u, v, w) = 1 - h_{0,0,1}^*(P, \mathcal{S})(1 + uv)w - h_{0,1,1}^*(P, \mathcal{S})(u + v)w + uvw^2.$$

When $n = 3$ and X° may be viewed as a family of non-compact, smooth surfaces, an explicit description of $E(X_\infty^\circ; u, v, w)$ is given by Theorem 1.5 and Example 4.13.

Example 1.7 Continuing with the explicit family of curves in Example 1.3, the corresponding Newton polytope P is the convex hull of $a_0 = (0, 0)$, $a_1 = (4, 0)$ and $a_2 = (0, 4)$. Setting $b_0 = (1, 1)$, $b_1 = (2, 1)$ and $b_2 = (1, 2)$, the lattice polyhedral subdivision \mathcal{S} has four maximal cells: $\{a_i, a_j, b_i, b_j\}$ for $i \neq j$ and $\{b_0, b_1, b_2\}$. By Example 1.6,

$$E(C_\infty^\circ; u, v, w) = -11 - 3(1 + uv)w + uvw^2.$$

In the case when we have a family of varieties over a punctured curve, we also give an alternative approach to Theorem 1.5 via intersection cohomology making use of the pure Hodge structure on the intersection cohomology of projective varieties. By the use of the decomposition theorem of Beilinson et al. [9], one can show that for certain stratifications, intersection cohomology admits a motivic formula if one includes terms accounting for the singularities in the normal cones to strata. This idea is used in the computation of intersection cohomology of toric varieties (see, e.g., [26]), in the work of Batyrev and Borisov [8], and is developed in greater generality by Cappell et al. [18]. Here, we observe that a motivic formula holds for the refined limit Hodge–Deligne polynomials for intersection cohomology with compact support (Theorem 6.1) and deduce that the following corollary is equivalent to Theorem 1.5 (see Lemma 6.2). The degree of $h^*(P, \mathcal{S}; u, v, w)$ as a polynomial in w is at most $\dim P + 1$, and we denote the coefficient of $w^{\dim P + 1}$ by $l^*(P, \mathcal{S}; u, v)$ and call it the **local limit mixed h^* -polynomial**.

Corollary 1.8 *Let $\mathbb{K} = \mathbb{C}(t)$ and let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, \mathcal{S}) and $\dim P = n$. Let X denote the closure of X° in the projective toric variety over \mathbb{K} corresponding to the normal fan of P . Then the refined limit Hodge–Deligne polynomial associated with the intersection cohomology of X is given by*

$$uvw^2 E_{\text{int}}(X_\infty; u, v, w) = uvw^2 E_{\text{int, Lef}}(P; uvw^2) + (-1)^{\dim P + 1} l^*(P, \mathcal{S}; u, v) w^{\dim P + 1},$$

where

$$(t - 1) E_{\text{int, Lef}}(P; t) = t^{\dim P} g([\emptyset, P]^*; t^{-1}) - g([\emptyset, P]^*; t)$$

is defined in terms of Stanley's g -polynomial (see Definition 4.3).

From the above corollary, one may deduce an explicit formula for the corresponding refined limit mixed Hodge numbers for intersection cohomology (see Corollary 6.3). When $\mathbb{K} = \mathbb{C}(t)$, we also present an alternative proof of Corollary 1.8 and hence of The-

orem 1.5 using intersection cohomology. This proof extends the ideas of Batyrev and Borisov's original proof of a formula for the usual mixed Hodge numbers of X_{gen}° in [8].

Example 1.9 As in Example 1.6, let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface. Let (P, S) denote the corresponding pair consisting of a lattice polytope together with a lattice polyhedral subdivision. Let X denote the closure of X° in the toric variety over \mathbb{K} corresponding to the normal fan of P . When $n = 2$, X may be viewed as a family of compact, smooth curves, and $E_{\text{int}}(X_\infty; u, v, w) = E(X_\infty; u, v, w)$ is computed in Example 1.6. When $n = 3$, X may be viewed as a family of compact, possibly singular surfaces, and $E_{\text{int}}(X_\infty; u, v, w)$ is given explicitly by Corollary 1.8, Example 4.13 and the computation $E_{\text{int}, \text{Lef}}(P; t) = 1 + \mu t + t^2$, where $\mu + 3$ is the number of facets of P .

We note that connections between limit mixed Hodge structures and the combinatorics of dual complexes have been studied in a number of contexts by Arapura et al. [1], Berkovich [11], Helm and the first author [33], Nicaise [46], and Payne [48].

1.1 Organization of the paper

This paper is structured as follows. In Sect. 2, we review necessary background from tropical geometry, introduce our invariant $\psi_{(X^\circ, \Sigma, \Delta)}$ of partial compactifications of subvarieties of algebraic tori, and prove Theorem 1.2. In Sect. 3, we discuss motivic invariants and the refined limit Hodge–Deligne polynomial. Section 4 introduces combinatorial invariants whose properties are established in [39] and which are related to the refined limit Hodge–Deligne polynomial of hypersurfaces of algebraic tori in Sect. 5. In Sect. 6, we derive a formula for the limit Hodge–Deligne polynomial of the intersection cohomology of a schön subvariety and use it to give an alternative proof of Theorem 1.5.

Notation and conventions. If \mathbb{P} is a toric variety, then we let $\mathbb{P}_{\mathbb{C}}$, $\mathbb{P}_{\mathbb{K}}$ and $\mathbb{P}_{\mathcal{O}}$ denote the corresponding toric variety over \mathbb{C} and \mathbb{K} , and corresponding toric scheme over \mathcal{O} , respectively.

2 A tropical approach to the motivic nearby fiber

In this section, we present the proof of Theorem 1.2. We will continue with the notation of the introduction. In particular, $X^\circ \subseteq (\mathbb{K}^*)^n$ is a schön subvariety, and Σ is a rational polyhedral structure on $\text{Trop}(X^\circ)$ that extends to a polyhedral subdivision of \mathbb{R}^n . Such a Σ exists by [43, Prop 6.8].

We first recall the following toric interpretation of the initial degenerations of X° , and refer the reader to [33, Section 1] for details. One can define a toric scheme $\mathbb{P}(\Sigma)_{\mathcal{O}}$ over \mathcal{O} from Σ . For a cell γ of Σ , let $\text{rec}(\gamma)$ denote the recession cone of γ . That is, $\text{rec}(\gamma)$ is the unique cone such that there exists a bounded polytope Q satisfying $\gamma = Q + \text{rec}(\gamma)$. By [17], the set of recession cones of Σ forms the recession fan Δ . Note that the bounded cells of Σ are precisely the cells whose recession cone is $\{0\}$. The generic fiber of $\mathbb{P}(\Sigma)_{\mathcal{O}}$ is the toric variety $\mathbb{P}(\Delta)_{\mathbb{K}}$. For cones τ in Δ , let U_τ be the corresponding torus orbit of $\mathbb{P}(\Delta)_{\mathbb{K}}$. Cells $\gamma \in \Sigma$ correspond to torus orbits U_γ contained in the central fiber of $\mathbb{P}(\Sigma)_{\mathcal{O}}$. We define T_γ to be the torus fixing U_γ pointwise.

Let \mathcal{X} denote the closure of X° in $\mathbb{P}(\Sigma)_{\mathcal{O}}$, and let X_Δ and X_0 denote the generic fiber and central fiber of \mathcal{X} , respectively. For cones τ in Δ , let $X_\tau^\circ = X_\Delta \cap U_\tau$, so that X_Δ admits

a stratification $X_\Delta = \cup_{\tau \in \Delta} X_\tau^\circ$. Similarly, for cells γ in Σ , if we let $X_\gamma^\circ = \mathcal{X} \cap U_\gamma$, then $X_0 = \cup_{\gamma \in \Sigma} X_\gamma^\circ$.

For each cone τ in Δ , let \mathbb{R}_τ denote the linear span of τ and consider the projection $\pi_\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n / \mathbb{R}_\tau$. Then X_τ° is a schön subvariety of U_τ , and its corresponding tropical variety has a polyhedral structure $\Sigma_\tau = \{\pi_\tau(\gamma) \mid \tau \subseteq \text{rec}(\gamma)\}$. In particular, the bounded cells of Σ_τ correspond to the cells of Σ with recession cone τ , and the recession fan Δ_τ of Σ_τ is the *star-quotient* of Δ by τ (see [28, Section 3.1]).

For w in the relative interior of γ , the initial degeneration $\text{in}_w X^\circ$ depends only on γ because the closure of X in $\mathbb{P}(\Sigma)_O$ is a tropical compactification by [43, Theorem 1.5]. Moreover, $\text{in}_w X^\circ$ is invariant under the torus T_γ . Moreover, there is a non-canonical isomorphism

$$\text{in}_w X^\circ \cong T_\gamma \times X_\gamma^\circ.$$

Then we have

$$[\text{in}_w X^\circ] = [T_\gamma][X_\gamma^\circ] = [(\mathbb{C}^*)^{\dim \gamma}][X_\gamma^\circ] = [\mathbb{C}^*]^{\dim \gamma}[X_\gamma^\circ] = (\mathbb{L} - 1)^{\dim \gamma}[X_\gamma^\circ]. \quad (3)$$

For any subfan Δ' of Δ , we define

$$\psi_{(X^\circ, \Sigma, \Delta')} := \sum_{\substack{\gamma \in \Sigma \\ \text{rec}(\gamma) \in \Delta'}} [X_\gamma^\circ](1 - \mathbb{L})^{\dim \gamma - \dim(\text{rec}(\gamma))}.$$

It follows from (3) that Theorem 1.2 is equivalent to the following:

$$\psi_{X^\circ} = \psi_{(X^\circ, \Sigma, \{0\})}. \quad (4)$$

It follows from the above description that for each cone τ in Δ ,

$$\psi_{(X_\tau^\circ, \Sigma_\tau, \{0\})} = \sum_{\substack{\gamma \in \Sigma \\ \text{rec}(\gamma) = \tau}} [X_\gamma^\circ](1 - \mathbb{L})^{\dim \gamma - \dim \tau}.$$

Hence, we have the relation

$$\psi_{(X^\circ, \Sigma, \Delta')} = \sum_{\tau \in \Delta'} \psi_{(X_\tau^\circ, \Sigma_\tau, \{0\})}. \quad (5)$$

A priori, $\psi_{(X^\circ, \Sigma, \{0\})}$ depends on Σ . Our first step is to show that is independent of Σ below. We first recall the following lemma that was proved in [38, Lemma 3.4]. We provide a more concise proof below.

Lemma 2.1 *Let P be a n -dimensional polytope and let Q be a proper (possibly empty) face of P . Let S be a polyhedral subdivision of P . Then*

$$\sum_{\substack{F \in S, F \cap Q = \emptyset \\ \text{Int}(F) \subseteq \text{Int}(P)}} (-1)^{\dim F} = \begin{cases} (-1)^d & \text{if } Q = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let P^\bullet and ∂P^\bullet be subcomplexes given as follows:

$$P^\bullet = \bigcup_{\substack{F \in S \\ F \cap Q = \emptyset}} F, \quad \partial P^\bullet = \bigcup_{\substack{F \in S, F \subset \partial P \\ F \cap Q = \emptyset}} F.$$

Now, the quantity on the left in the statement is the relative Euler characteristic $\chi(P^\bullet, \partial P^\bullet)$. If $Q = \emptyset$, this becomes $\chi(P, \partial P) = \chi(B^n, S^{n-1})$ in which case the theorem holds.

Now suppose that $Q \neq \emptyset$. It suffices to show that the inclusion $\partial P^\bullet \hookrightarrow P^\bullet$ induces an isomorphism in homology. Consider the commutative diagram

$$\begin{array}{ccc} \partial P^\bullet & \hookrightarrow & P^\bullet \\ \downarrow & & \downarrow \\ \partial P \setminus Q & \hookrightarrow & P \setminus Q \end{array}$$

The inclusion $P^\bullet \hookrightarrow P \setminus Q$ is a homotopy equivalence. We give its homotopy inverse. Let

$$P_i^\bullet = P^\bullet \cup \bigcup_{\substack{F \in \mathcal{S} \\ \dim(F) \leq i, F \cap Q \neq \emptyset}} (F \setminus Q).$$

Let $r_i : P_{i+1}^\bullet \rightarrow P_i^\bullet$ be given as the identity on cells F disjoint from Q and given by $r_F : F \setminus Q \rightarrow \partial F \setminus Q$ for cells intersecting Q where r_F is projection away from some point x_F in the relative interior of $F \cap Q$. We define $r : P \setminus Q \rightarrow P^\bullet$ to be the composition $r_0 \circ \dots \circ r_d : P \setminus Q = P_d^\bullet \rightarrow P_0^\bullet = P^\bullet$.

The inclusion $\partial P^\bullet \hookrightarrow \partial P \setminus Q$ is also a homotopy equivalence. Its inverse is defined similarly to the map above. Finally, $\partial P \setminus Q \hookrightarrow P \setminus Q$ is a homotopy equivalence whose inverse can be given by projection from a point in the relative interior of Q . Since these three maps induce isomorphisms in homology, so must $\partial P^\bullet \hookrightarrow P^\bullet$. \square

The following lemma is analogous to [38, Theorem 3.6].

Lemma 2.2 *The expression $\psi_{(X^\circ, \Sigma, \{0\})}$ above is independent of the choice of rational polyhedral structure Σ on $\text{Trop}(X^\circ)$.*

Proof Suppose Σ' is a rational polyhedral structure on $\text{Trop}(X^\circ)$ corresponding to a toric scheme $\mathbb{P}(\Sigma')_{\mathcal{O}}$. After taking a common refinement, we can suppose that Σ' is a refinement of Σ . This induces a proper morphism of toric schemes $\mathbb{P}(\Sigma')_{\mathcal{O}} \rightarrow \mathbb{P}(\Sigma)_{\mathcal{O}}$. If γ' is a cell of Σ' , then the relative interior of γ' lies in the relative interior $\text{Int}(\gamma)$ of a unique cell γ of Σ . By standard toric geometry [28, Sec 2.1], the corresponding morphism of tori $U_{\gamma'} \rightarrow U_\gamma$ factors as

$$U_{\gamma'} \cong U_\gamma \times (\mathbb{C}^*)^{\dim \gamma - \dim \gamma'} \rightarrow U_\gamma,$$

where the second map is projection onto the first coordinate. By [43, Proposition 7.6], $X_{\gamma'}^\circ$ is the pullback of X_γ° , and hence $[X_{\gamma'}^\circ] = [X_\gamma^\circ](\mathbb{L} - 1)^{\dim \gamma - \dim \gamma'}$. We compute

$$\begin{aligned} & \sum_{\substack{\gamma' \in \Sigma' \\ \gamma' \text{ bounded}}} [X_{\gamma'}^\circ] (1 - \mathbb{L})^{\dim \gamma'} \\ &= \sum_{\gamma \in \Sigma} [X_\gamma^\circ] (1 - \mathbb{L})^{\dim \gamma} \left(\sum_{\substack{\gamma' \in \Sigma', \gamma' \text{ bounded} \\ \text{Int}(\gamma') \subseteq \text{Int}(\gamma)}} (-1)^{\dim \gamma - \dim \gamma'} \right). \end{aligned}$$

Hence, it is enough to show that

$$\sum_{\substack{\gamma' \in \Sigma', \gamma' \text{ bounded} \\ \text{Int}(\gamma') \subseteq \text{Int}(\gamma)}} (-1)^{\dim \gamma'} = \begin{cases} (-1)^{\dim \gamma} & \text{if } \gamma \text{ is bounded} \\ 0 & \text{otherwise.} \end{cases}$$

This follows directly from Lemma 2.1 if we do the following: Let C_γ be the cone over $\gamma \times 1$ in $\mathbb{R}^n \times \mathbb{R}$; choose H to be an affine hyperplane such that $P = C_\gamma \cap H$ is a polytope not containing the origin; set Q to be the intersection of P with $\mathbb{R}^n \times \{0\}$; and let \mathcal{S} be polyhedral subdivision of P induced by the fan refinement of C_γ induced by Σ' . The cells in \mathcal{S} that intersect Q correspond to unbounded cells in Σ' . \square

Steenbrink has applied a result [58, Theorem 5] similar to Lemma 2.2 to study motivic Milnor fibers of function germs on toric singularities.

Since $\psi_{(X^\circ, \Sigma, \{0\})}$ is independent of the choice of Σ by Lemma 2.2, after possible ramified base-extension of \mathbb{K} , it follows from [33, Proposition 2.3] that we may choose Σ such that $\mathbb{P}(\Delta)_\mathbb{K}$ is smooth. In this case, we may invoke the following result.

Theorem 2.3 [38, Theorem 5.1] *With the notation above, if $\mathbb{P}(\Delta)_\mathbb{K}$ is smooth, then the motivic nearby fiber ψ_{X_Δ} of X_Δ is equal to $\psi_{(X^\circ, \Sigma, \Delta)}$.*

By Theorem 2.3, (5), and induction on dimension, we have

$$\psi_{X_\Delta} = \psi_{(X^\circ, \Sigma, \Delta)} = \sum_{\tau \in \Delta} \psi_{(X_\tau^\circ, \Sigma_\tau, \{0\})} = \psi_{(X^\circ, \Sigma, \{0\})} + \sum_{\{0\} \neq \tau \in \Delta} \psi_{X_\tau^\circ}.$$

Since $X_\Delta = \bigcup_{\tau \in \Delta} X_\tau^\circ$, the additivity of the motivic nearby fiber and the above expression imply that

$$\psi_{X^\circ} = \psi_{X_\Delta} - \sum_{\{0\} \neq \tau \in \Delta} \psi_{X_\tau^\circ} = \psi_{(X^\circ, \Sigma, \{0\})}.$$

This completes the proof of (4) and hence Theorem 1.2.

Using (5), we immediately deduce the following corollary.

Corollary 2.4 *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be schön, and let Σ be a rational polyhedral structure on $\text{Trop}(X^\circ)$ extending to a polyhedral subdivision of \mathbb{R}^n . For each cell γ of Σ , let $\text{rec}(\gamma)$ denote the corresponding recession cone. For any subfan Δ' of the recession fan Δ of Σ , let $X_{\Delta'}$ denote the closure of X° in the corresponding toric variety $\mathbb{P}(\Delta')_\mathbb{K}$. Then the motivic nearby fiber $\psi_{X_{\Delta'}}$ is given by:*

$$\psi_{X_{\Delta'}} = \sum_{\substack{\gamma \in \Sigma \\ \text{rec}(\gamma) \in \Delta'}} [X_\gamma^\circ] (1 - \mathbb{L})^{\dim \gamma - \dim(\text{rec}(\gamma))}.$$

Remark 2.5 Note that when $\Delta' = \{0\}$ above, we recover Theorem 1.2, while when $\Delta' = \Delta$, then we recover the statement of Theorem 2.3 without the assumption that $\mathbb{P}(\Delta)_\mathbb{K}$ is smooth. In this way, we see that Theorem 1.2 is a generalization of Theorem 2.3. Note that X_Δ is proper, and, while the assumption that $\mathbb{P}(\Delta)_\mathbb{K}$ is smooth forces X_Δ to be smooth, in general, X_Δ and $\mathbb{P}(\Delta)_\mathbb{K}$ may have singularities.

Remark 2.6 Note that Theorem 1.2 implies that if X° is schön, then the expression $\psi_{(X^\circ, \Sigma, \{0\})}$ is not only independent of Σ (Lemma 2.2), but independent of the choice of embedding $X^\circ \subseteq (\mathbb{K}^*)^n$. We do not know a direct proof of this fact.

Remark 2.7 As in [38, Section 3], the definition of $\psi_{(X^\circ, \Sigma, \{0\})}$ can be extended to the case when X° is not necessarily schön, but the pair $(X^\circ, \mathbb{P}(\Sigma)_\mathcal{O})$ is *tropical*. In this case, the proof of Lemma 2.2 holds unchanged and $\psi_{(X^\circ, \Sigma, \{0\})}$ is independent of the choice of Σ . The expression $\psi_{(X^\circ, \Sigma, \Delta)}$ was called the *tropical motivic nearby fiber* in [38]. However,

one cannot expect an analog of Theorem 1.2, as the following example demonstrates. We also do not know how the tropical motivic nearby fiber depends on the embedding of X° into $(\mathbb{K}^*)^n$.

Let $G(X, Y, Z)$ be a homogeneous polynomial over \mathbb{C} of degree 3 whose zero locus $V(G)$ in \mathbb{P}^2 is a nodal cubic curve. Suppose further that G has the following properties

- (1) all coefficients of degree 3 monomials in G are nonzero,
- (2) the node of $V(G)$ lies in $(\mathbb{C}^*)^2 \subset \mathbb{P}^2$, and
- (3) $V(G)$ intersects each coordinate lines in three distinct points.

It is possible to find such a G by applying a generic element of $\mathrm{Gl}_3(\mathbb{C})$ to the equation of a nodal cubic. The tropicalization of $V(G)^\circ \subseteq (\mathbb{C}^*)^2$ in \mathbb{R}^2 consists of the origin and three rays in the directions $(1, 0)$, $(0, 1)$, $(-1, -1)$, each with multiplicity 3. Now let H be a generic homogeneous polynomial of degree 3. Consider $F = G + tH$ considered as a homogeneous polynomial over \mathbb{K} . Now, $V(F)$ is a smooth cubic over \mathbb{K} . Consequently, for $t \neq 0$ sufficiently small, $V(F)_{\mathrm{gen}}^\circ$ is a smooth cubic over \mathbb{C} with 9 points removed. By construction, the tropicalization of $V(F)^\circ \subseteq (\mathbb{K}^*)^2$ is the same as that of $V(G)^\circ$. Moreover, $(V(F)^\circ, \mathbb{P}(\Sigma)_\circ)$ is tropical, where Σ is the standard polyhedral structure on the tropicalization of $V(F)^\circ$. There is a single-bounded cell which is the origin. Since $\mathrm{in}_0 V(F)^\circ = V(G)^\circ$, we have $\psi_{(V(F)^\circ, \Sigma, \{0\})} = [V(G)^\circ]$. This is a nodal cubic minus the 9 points of intersection with the coordinate lines. Because a nodal cubic is isomorphic to a projective line with two points identified, we have $e(V(G)^\circ) = -8 \neq e(V(F)_{\mathrm{gen}}^\circ) = -9$ violating the final statement in Corollary 1.4.

3 The motivic nearby fiber and limit mixed Hodge structures

As observed in the introduction, by composing the motivic nearby fiber (1) with a motivic invariant over \mathbb{C} , we obtain a motivic invariant over \mathbb{K} to which we can apply Theorem 1.2. In this section, we introduce some known results from the theory of limit mixed Hodge structures. We recommend [49] and [50] as references. The theory was developed by many authors including Deligne et al. [19], Schmid [54], Steenbrink [57], and Saito [53].

Throughout this section, if a complex vector space B admits a mixed Hodge structure (W_\bullet, F^\bullet) [50] with associated graded pieces

$$B^{p,q} \cong \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(B),$$

then we write $h^{p,q}(B) = \dim B^{p,q}$. For a sequence of such vector spaces $B_\bullet = \{B_m \mid m \geq 0\}$, set $e^{p,q}(B_\bullet) = \sum_m (-1)^m h^{p,q}(B_m)$. Then the *Hodge polynomial* of B_\bullet is defined by

$$E(B_\bullet) = E(B_\bullet; u, v) = \sum_{p,q} e^{p,q}(B_\bullet) u^p v^q.$$

3.1 Motivic invariants over \mathbb{C}

In [21], Deligne proved that the m th cohomology group with compact supports $H_c^m(V)$ of a complex variety V admits a canonical mixed Hodge structure with decreasing filtration F^\bullet called the **Hodge filtration** and increasing filtration W_\bullet called the **Deligne weight filtration**. The set of numbers $\{h^{p,q}(H_c^m(V))\}_{p,q,m}$ is called the **mixed Hodge numbers** of V , and the corresponding Hodge polynomial $E(V; u, v)$ is the **Hodge–Deligne polynomial** of V . The corresponding motivic invariant over \mathbb{C} is the *Hodge–Deligne map*

$$E : K_0(\mathrm{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}[u, v], [V] \mapsto E(V; u, v).$$

The Hodge–Deligne polynomial of V specializes to the χ_y -characteristic $E(V; u, 1)$ of V . Its coefficients are alternating sums of the dimensions of the graded pieces of the Hodge filtration on the cohomology of V with compact supports. The **Euler characteristic** $e(V)$ is obtained via the specialization $e(V) = E(V; 1, 1)$.

Example 3.1 Recall that we write $\mathbb{L} := [\mathbb{A}^1]$ in the Grothendieck ring $K_0(\text{Var}_k)$ for any field k . The complex affine line has $h^{1,1}(H_c^2(\mathbb{A}^1)) = 1$ and all other mixed Hodge numbers equal to zero. Hence its Hodge–Deligne polynomial is $E(\mathbb{A}^1) = uv$.

The n -dimensional torus $(\mathbb{C}^*)^n$ has $[(\mathbb{C}^*)^n] = (\mathbb{L} - 1)^n$ in $K_0(\text{Var}_{\mathbb{C}})$, $E((\mathbb{C}^*)^n) = (uv - 1)^n$, and its mixed Hodge numbers are all zero except

$$h^{k,k} \left(H_c^{n+k}((\mathbb{C}^*)^n) \right) = \binom{n}{k},$$

for all k .

3.2 Motivic invariants over \mathbb{K}

Recall from the introduction that we regard a variety X over \mathbb{K} as a family of complex varieties over the disk \mathbb{D}^* , and we fix a nonzero fiber $X_{\text{gen}} := f^{-1}(t)$ for some $t \in \mathbb{D}^*$. Then the cohomology groups $H_c^m(X_{\text{gen}})$ admit a weight filtration M_{\bullet} called the **monodromy weight filtration**. We will write $H_c^m(X_{\infty})$ to denote $H_c^m(X_{\text{gen}})$ with the mixed Hodge structure $(F^{\bullet}, M_{\bullet})$. The corresponding mixed Hodge numbers are denoted $h^{p,q}(H_c^m(X_{\infty}))$ and are called the **limit mixed Hodge numbers** of X . The corresponding Hodge polynomial is denoted $E(X_{\infty}; u, v)$ and is called the **limit Hodge–Deligne polynomial** of X . It is the composition of the motivic nearby fiber and the Hodge–Deligne map, i.e., $E(X_{\infty}; u, v) = E(\psi_X)$. It specializes to both the χ_y -characteristic of X_{gen} and the Euler characteristic of X_{gen} (see Remark 3.5 below).

The monodromy weight filtration encodes the action of the logarithm of the monodromy operator on the W_{\bullet} -graded pieces of $H_c^m(X_{\text{gen}})$. We explain this statement in detail below.

The cohomology groups $H_c^m(X_t)$ of the fibers are isomorphic as vector spaces but have a Hodge structure which varies. Because $H_c^m(X_t)$ forms a locally trivial fiber bundle, parallel transport gives a monodromy transformation, $T : H_c^m(X_{\text{gen}}) \rightarrow H_c^m(X_{\text{gen}})$. It turns out that T is quasi-unipotent, that is, some multiple of T is unipotent. After replacing T by some power which corresponds to pulling back the family by a map $\mathbb{D}^* \rightarrow \mathbb{D}^*$ ramified over the puncture, we may take its logarithm, $N = \log T$, and obtain a nilpotent operator. Moreover, the monodromy map T preserves the weight filtration W_{\bullet} , and $N : H_c^m(X_{\infty}) \rightarrow H_c^m(X_{\infty})$ is a morphism of mixed Hodge structures of type $(-1, -1)$.

It follows that for every nonnegative integer r , N restricts to a nilpotent operator $N(r)$ on the graded piece $Gr_r^W H_c^m(X_{\text{gen}})$. Let $F(r)^{\bullet}$ and $M(r)_{\bullet}$ denote the filtrations on $Gr_r^W H_c^m(X_{\text{gen}})$ induced by F^{\bullet} and M_{\bullet} , respectively. Then $N(r)^{r+1} = 0$ and $M(r)_{\bullet}$ is the filtration obtained from $N(r)$ which determines and is determined by the Jordan block decomposition of $N(r)$. Indeed, we may inductively define a unique increasing filtration

$$0 \subseteq M(r)_0 \subseteq M(r)_1 \subseteq \cdots \subseteq M(r)_{2r} = Gr_r^W H_c^m(X_{\text{gen}})$$

satisfying the following properties for any nonnegative integer k ,

- (1) $N(r)(M(r)_k) \subseteq M(r)_{k-2}$,
- (2) the induced map $N(r)^k : Gr_{r+k}^{M(r)} Gr_r^W H_c^m(X_{\text{gen}}) \rightarrow Gr_{r-k}^{M(r)} Gr_r^W H_c^m(X_{\text{gen}})$ is an isomorphism.

The pair $(F(r)^\bullet, M(r)_\bullet)$ determines a limit mixed Hodge structure on $Gr_r^W H_c^m(X_{\text{gen}})$. Moreover, $N(r)$ is a morphism of mixed Hodge structures of type $(-1, -1)$. We will write $Gr_r^W H_c^m(X_\infty)$ when referring to $Gr_r^W H_c^m(X_{\text{gen}})$ with the limit mixed Hodge structure. We denote the corresponding mixed Hodge numbers by $h^{p,q,r}(H_c^m(X_\infty))$ and call them the **refined limit mixed Hodge numbers** of X . For each r , we encode the corresponding Hodge polynomial as the coefficient of w^r in a polynomial $E(X_\infty; u, v, w)$ that we will call the **refined limit Hodge–Deligne polynomial**.

Remark 3.2 It follows from Saito’s theory of mixed Hodge modules [53] (see [4] for a survey) that most natural morphisms between varieties X over \mathbb{K} give rise to morphisms of complex varieties X_{gen} that respect the filtrations $(F^\bullet, W_\bullet, M_\bullet)$. In particular, if $U \subseteq X$ is an open inclusion and $V = X \setminus U$, then the corresponding long exact sequence of cohomology with compact supports for the triple $(X_{\text{gen}}, U_{\text{gen}}, V_{\text{gen}})$ consists of morphisms that preserve the Hodge filtration and both the Deligne and monodromy weight filtrations (c.f. proof of [50, Lemma 14.61], see also [27] for the classical approach). In particular, it follows from Remark 3.4 that the refined limit Hodge–Deligne polynomial is a motivic invariant over \mathbb{K} . That is, we may consider the *refined Hodge–Deligne map*

$$\bar{E} : K_0(\text{Var}_{\mathbb{K}}) \rightarrow \mathbb{Z}[u, v, w], [X] \mapsto E(X_\infty; u, v, w).$$

3.3 Properties of the refined limit Hodge–Deligne polynomial

We now collect some of the basic properties of the above motivic invariants over \mathbb{K} . In particular, we will give a complete description of the following commutative diagram of ring homomorphisms

$$\begin{array}{ccccc} K_0(\text{Var}_{\mathbb{K}}) & \xrightarrow{\bar{E}} & \mathbb{Z}[u, v, w] & \xrightarrow[u \mapsto uw^{-1}]{v \mapsto 1} & \mathbb{Z}[u, w] \\ \downarrow \psi & & \downarrow w \mapsto 1 & & \downarrow w \mapsto 1 \\ K_0(\text{Var}_{\mathbb{C}}) & \xrightarrow{E} & \mathbb{Z}[u, v] & \xrightarrow[v \mapsto 1]{} & \mathbb{Z}[u] \xrightarrow[u \mapsto 1]{} \mathbb{Z}. \end{array}$$

Remark 3.3 The refined limit mixed Hodge numbers have the following explicit description in terms of $(F^\bullet, W_\bullet, M_\bullet)$,

$$h^{p,q,r}(H_c^m(X_\infty)) = \dim(Gr_{F(r)}^p Gr_{p+q}^{M(r)} Gr_r^W H_c^m(X_\infty)).$$

If we sum over r , we discard the refinement by the weight filtration and obtain the following relation with the limit mixed Hodge numbers,

$$h^{p,q}(H_c^m(X_\infty)) = \dim(Gr_F^p Gr_{p+q}^M H_c^m(X_\infty)) = \sum_r h^{p,q,r}(H_c^m(X_\infty)). \quad (6)$$

Similarly, if we sum over q , we discard the monodromy filtration refinement and obtain mixed Hodge numbers of X_{gen} ,

$$h^{p,r-p}(H_c^m(X_{\text{gen}})) = \sum_q h^{p,q,r}(H_c^m(X_\infty)). \quad (7)$$

Summing the refined limit mixed Hodge numbers over q and r gives the following relation between the limit mixed Hodge numbers and the mixed Hodge numbers of X_{gen} ,

$$\sum_q h^{p,q}(H_c^m(X_\infty)) = \sum_q h^{p,q}(H_c^m(X_{\text{gen}})). \quad (8)$$

Remark 3.4 The refined limit Hodge–Deligne polynomial is described explicitly in terms of the refined limit mixed Hodge numbers as

$$E(X_\infty; u, v, w) = \sum_{p,q,r} e^{p,q,r}(X_\infty) u^p v^q w^r,$$

where

$$e^{p,q,r}(X_\infty) = e^{q,p,r}(X_\infty) = \sum_m (-1)^m h^{p,q,r}(H_c^m(X_\infty)).$$

Using (6) and (7), we see that the refined limit Hodge–Deligne polynomial specializes to both the limit Hodge–Deligne polynomial and the Hodge–Deligne polynomial of X_{gen} ,

$$\begin{aligned} E(X_\infty; u, v) &= E(X_\infty; u, v, 1), \\ E(X_{\text{gen}}; u, w) &= E(X_\infty; u w^{-1}, 1, w). \end{aligned}$$

Remark 3.5 We see from Remark 3.4 that the limit Hodge–Deligne polynomial specializes to both the χ_Y -characteristic of X_{gen} ,

$$E(X_\infty; u, 1) = E(X_{\text{gen}}; u, 1), \quad (9)$$

and the Euler characteristic of X_{gen}

$$E(X_\infty; 1, 1) = e(X_{\text{gen}}),$$

Remark 3.6 With the notation above, since $N(r)$ is a morphism of mixed Hodge structures of type $(-1, -1)$, the isomorphisms (2) imply that each vertical strip of the Hodge diamond of $Gr_r^W H_c^m(X_\infty)$ is a symmetric, unimodal sequence of nonnegative integers. That is, for $0 \leq k \leq r$, the sequence $\{h^{k+i, i, r}(H_c^m(X_\infty)) \mid 0 \leq i \leq r-k\}$ is symmetric and unimodal.

Remark 3.7 By construction, the refined limit mixed Hodge numbers are symmetric in p and q , i.e., $h^{p,q,r}(H_c^m(X_\infty)) = h^{q,p,r}(H_c^m(X_\infty))$. It follows from Remark 3.6 that they satisfy the additional symmetry:

$$h^{p,q,r}(H_c^m(X_\infty)) = h^{r-p, r-q, r}(H_c^m(X_\infty)).$$

In particular, the refined limit Hodge–Deligne polynomial satisfies the symmetries

$$\begin{aligned} E(X_\infty; u, v, w) &= E(X_\infty; v, u, w), \\ E(X_\infty; u, v, w) &= E(X_\infty; u^{-1}, v^{-1}, uvw). \end{aligned}$$

Example 3.8 If V is a complex variety and $X = V \times_{\mathbb{C}} \mathbb{K}$, then X may be regarded as a trivial family over \mathbb{D}^* . In this case, N is identically zero, M_\bullet coincides with the Deligne filtration W_\bullet , and $E(X_\infty; u, v) = E(V; u, v)$. Moreover,

$$E(X_\infty; u, v, w) = E(V; uw, vw).$$

Example 3.9 If X is smooth and proper, then $Gr_r^W H^m(X_{\text{gen}}) = 0$ unless $m = r$. In this case, the monodromy weight filtration encodes the Jordan block decomposition of $N : H^m(X_\infty) \rightarrow H^m(X_\infty)$. Moreover,

$$E(X_\infty; u, v, w) = \sum_{p,q,m} (-1)^m h^{p,q}(H^m(X_\infty)) u^p v^q w^m.$$

Remark 3.10 We claim that if there exists a function $\nu : \mathbb{Z} \rightarrow \mathbb{Z}$, such that $H^{p,q}(H_c^m(X_{\text{gen}})) = 0$ for $p \neq \nu(m)$, then $N = 0$. Indeed, by (8),

$$\sum_q h^{p,q}(H_c^m(X_\infty)) = \sum_q h^{p,q}(H_c^m(X_{\text{gen}})) = 0$$

for $p \neq \nu(m)$. Since $N : H_c^m(X_\infty) \rightarrow H_c^m(X_\infty)$ is a morphism of mixed Hodge structures of type $(-1, -1)$, for any (p, q) , either the source or target of the induced map $N : H^{p,q}(H_c^m(X_\infty)) \rightarrow H^{p-1,q-1}(H_c^m(X_\infty))$ is zero.

Example 3.11 By Example 3.1 and Remark 3.10, if $X_{\text{gen}} \cong (\mathbb{C}^*)^n$, then $N = 0$. Hence, by Example 3.8,

$$E(X_\infty; u, v, w) = E((\mathbb{K}^*)^n; u, v, w) = E((\mathbb{C}^*)^n; uw, vw) = (uvw^2 - 1)^n.$$

4 Subdivisions of lattice polytopes

In this section, we gather together the relevant facts that we will need about the combinatorics of subdivisions of polytopes. Details and proofs of all statements can be found in [39]. We say that the empty polytope has dimension -1 .

A **polyhedral subdivision** of a polytope $P \subset \mathbb{R}^n$ is a subdivision of P into a finite number of polytopes such that the intersection of any two polytopes is a (possibly empty) face of both. A **lattice polyhedral subdivision** of a lattice polytope P is a polyhedral subdivision of P into lattice polytopes. A natural class of polyhedral subdivisions is the regular subdivisions. They are induced by a height function $\omega : P \cap \mathbb{Z}^n \rightarrow \mathbb{R}$. The cells of the subdivision are the projections of the bounded faces of the convex hull of $\text{UH} = \{(u, \lambda) \mid \lambda \geq \omega(u)\} \in \mathbb{R}^n \times \mathbb{R}$. A subdivision is said to be **regular** if it is induced by some height function. For more details, see [22, 29].

We first recall some definitions concerning the combinatorics of Eulerian posets. Consider a finite poset B containing a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$. For any pair $z \leq x$ in B , we can consider the interval $[z, x] = \{y \in B \mid z \leq y \leq x\}$. Assume that B is *graded* in the sense that for every $x \in B$, every maximal chain in the interval $[\widehat{0}, x]$ has the same length $\rho(x)$. We call $\rho : B \rightarrow \mathbb{N}$ the *rank function* of B , and call $\rho(\widehat{1})$ the *rank* of B . Then B is **Eulerian** if every interval $[z, x]$ with $z < x$ has as many elements of odd rank as even rank.

Example 4.1 The poset of faces of a polytope P (including the empty face) is an Eulerian poset under inclusion. Then $\rho(Q) = \dim Q + 1$, for any face Q of P . Let \mathcal{S} be a polyhedral subdivision of P , and let F be a (possibly empty) cell of \mathcal{S} . As a poset, the **link** $\text{lk}_{\mathcal{S}}(F)$ of F in \mathcal{S} consists of all cells F' of \mathcal{S} that contain F under inclusion, and we have that the interval $[F, F']$ is an Eulerian poset.

Example 4.2 If B is a poset, then B^* is the poset with the same elements as B and all orderings reversed. In particular, B is Eulerian if and only if B^* is Eulerian.

The **g -polynomial** of an Eulerian poset is defined recursively and was introduced by Stanley [55, Corollary 6.7].

Definition 4.3 Let B be an Eulerian poset of rank n . If $n = 0$, then $g(B; t) = 1$. If $n > 0$, then $g(B; t)$ is the unique polynomial of degree strictly less than $n/2$ satisfying

$$t^n g(B; t^{-1}) = \sum_{x \in B} (t-1)^{n-\rho(x)} g([\widehat{0}, x]; t).$$

The following theorem of Stanley giving an inversion formula for g is used in Sect. 6:

Theorem 4.4 [55, Corollary 8.3] *If B is Eulerian and has positive rank, then*

$$\sum_{x \in B} (-1)^{\rho(x)} g([\widehat{0}, x]; t) g([x, \widehat{1}]^*; t) = \sum_{x \in B} (-1)^{\rho(x)} g([\widehat{0}, x]^*; t) g([x, \widehat{1}]; t) = 0.$$

We will be interested in the following example [55, Example 7.2].

Example 4.5 Let \mathcal{S} be a polyhedral subdivision of a polytope P . If F is a (possibly empty) cell of \mathcal{S} , then the h -**polynomial** of $\text{lk}_{\mathcal{S}}(F)$ is defined by

$$t^{\dim P - \dim F} h(\text{lk}_{\mathcal{S}}(F); t^{-1}) = \sum_{\substack{F' \in \mathcal{S} \\ F \subseteq F'}} (t-1)^{\dim P - \dim F'} g([F, F']; t).$$

We now recall some basic Ehrhart theory. Let P be a non-empty lattice polytope in a lattice M of rank n . For $m \in \mathbb{Z}_{>0}$, consider the function $f_P(m) = \#(mP \cap M)$. By Ehrhart's theorem [10, Section 3.3], $f_P(m)$ is a polynomial of degree $\dim P$, called the **Ehrhart polynomial** of P . It follows that we can write

$$f_P(m) = f_0(P) + f_1(P)m + \cdots + f_n(P)m^n, \quad (10)$$

where $f_i(P) \in \mathbb{Z}$, and

$$1 + \sum_{m>0} f_P(m)u^m = \frac{h^*(P; u)}{(1-u)^{\dim P+1}}.$$

where $h^*(P; u)$ of P is a polynomial of degree at most $\dim P$ called the h^* -**polynomial** of P (see, e.g., [10, Section 3.3]). Note that if P is empty, then we set $f_P(m) \equiv 0$ and $h^*(P; u) = 1$. We have $h^*(P; 1) = (\dim P)! \text{vol}(P)$ where $\text{vol}(P)$ is the Euclidean volume of P .

These invariants play the central role in the theory of valuations on polytopes. The definition given below is a priori weaker than the usual definition of valuations but is equivalent as a consequence of Lemma 4.7.

Definition 4.6 Let \mathcal{P}_M be the set of lattice polytopes for a lattice M and let G be a group. A G -valued **valuation** on \mathcal{P}_M is a map $\varphi : \mathcal{P}_M \rightarrow G$ satisfying

- (1) If \mathcal{S} is a regular lattice subdivision of P with top-dimensional cells P_1, \dots, P_m , φ satisfies the inclusion/exclusion relation

$$\varphi(P) = \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} (-1)^{\dim P - \dim F} \varphi(F),$$

- (2) $\varphi(\emptyset) = 0$, and
- (3) $\varphi(P) = \varphi(UP + u)$ for $P \in \mathcal{P}_M$, $U \in \text{Aut}(M)$, $u \in M$.

The lemma below is non-trivial since not every lattice polytope admits a lattice polyhedral subdivision into unimodular simplices, let alone a regular one. This lemma is an adaptation of [31, Prop 19.2] which is stated for general lattice subdivisions.

Lemma 4.7 *Valuations are determined by their values on unimodular simplices: If φ_1, φ_2 are valuations that are equal on unimodular simplices, then $\varphi_1 = \varphi_2$.*

Proof Let \mathcal{G} be the free Abelian group generated by convex lattice polytopes in M . Let \mathcal{H} be the subgroup generated by the following:

- (1) For \mathcal{S} is a regular subdivision of P ,

$$P - \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} (-1)^{\dim P - \dim F} \varphi(F),$$

- (2) \emptyset ,

- (3) $P - (UP + u)$ for $P \in \mathcal{P}_{\mathbb{Z}^n}$, $U \in \text{Aut}(M)$, $u \in M$

We show that \mathcal{G}/\mathcal{H} is generated by unimodular simplices. We induct on the dimension of M . Let \mathcal{H}' be the subgroup of \mathcal{G} generated by convex lattice polytopes of M whose affine span is not full-dimensional. By induction, we may suppose $\mathcal{H}'/(\mathcal{H}' \cap \mathcal{H})$ is generated by unimodular simplices. Now, it suffices to show that $\mathcal{G}/(\mathcal{H} + \mathcal{H}')$ is generated by a d -dimensional unimodular simplex.

First, every polytope has a regular triangulation (see, e.g., [22, Proposition 2.2.4]). Therefore, every polytope in $\mathcal{G}/(\mathcal{H} + \mathcal{H}')$ can be written as a formal sum of lattice simplices. It remains to show that every lattice simplex is equal in $\mathcal{G}/(\mathcal{H} + \mathcal{H}')$ to a formal multiple of a unimodular simplex. We induct on the volume of the lattice simplex. Let $P \subset M_{\mathbb{R}}$ be a d -dimensional lattice simplex. If $\text{vol}(P) = 1$ then we are done. Suppose $\text{vol}(P) = V \geq 2$, and let F_0, \dots, F_d denote the facets of P . By the proof of [31, Proposition 19.1], there is a point $p \in M$ such that $\text{vol}(\text{Conv}(F_i \cup \{p\})) < V$. Let $\omega : \text{Vert}(P) \cup \{p\} \rightarrow \mathbb{R}$ be the height function that is 0 on the vertices of P and 1 on p . The graph of the height function lies in $M_{\mathbb{R}} \times \mathbb{R}$, and because the points in the graph are affinely independent, their convex hull is a simplex. The projections of the convex hull by $\pi : M \times \mathbb{R} \rightarrow M$ are $\text{Conv}(P \cup \{p\})$. Moreover, the projections of the upper faces or the lower faces each give regular subdivisions $\mathcal{S}_{\text{upper}}, \mathcal{S}_{\text{lower}}$ of $\text{Conv}(P \cup \{p\})$. The top-dimensional lower faces of the convex hull are P and some faces $\text{Conv}(F_i \cup \{p\})$ for $i \in I$ for some subset $I \subset \{0, \dots, n\}$. The top-dimensional upper faces of the convex hull are $\text{Conv}(F_i \cup \{p\})$ for $i \notin I$. The subdivision relation in $\mathcal{G}/(\mathcal{H} + \mathcal{H}')$ gives

$$\text{Conv}(P \cup \{p\}) = P + \sum_{i \in I} \text{Conv}(F_i \cup \{p\}) = \sum_{i \notin I} \text{Conv}(F_i \cup \{p\}).$$

Therefore, we have in $\mathcal{G}/(\mathcal{H} + \mathcal{H}')$,

$$P = \sum_{i \in I} \text{Conv}(F_i \cup \{p\}) - \sum_{i \notin I} \text{Conv}(F_i \cup \{p\}).$$

This gives an expression for P as a formal sum of simplices of smaller volume. \square

Since $P \mapsto f_P(m)$ is a $\mathbb{Z}[m]$ -valued valuation, each $f_i : P \mapsto f_i(P)$ is a \mathbb{Z} -valued valuation for $i = 0, \dots, n$. For example,

$$f_0(P) = \begin{cases} 1 & \text{if } P \neq \emptyset \\ 0 & \text{if } P = \emptyset. \end{cases}$$

By the Betke–Kneser theorem ([12], [31, Theorem 19.6]), $\{f_0, \dots, f_n\}$ are a \mathbb{Z} -basis for the group of all \mathbb{Z} -valued valuations of \mathcal{P}_M .

Example 4.8 For a fixed nonnegative integer m , we have a valuation

$$P \mapsto f_P(m) = f_0(P) + f_1(P)m + \dots + f_n(P)m^n.$$

The **local h^* -polynomial** $l^*(P; u)$ of P was introduced by Stanley in [55, Example 7.13], generalizing the definition of Betke and McMullen in the case of a simplex [13], and was independently introduced by Borisov and Mavlyutov in [15],

$$l^*(P; u) = \sum_{Q \subseteq P} (-1)^{\dim P - \dim Q} h^*(Q; u) g([Q, P]^*; u).$$

Our main combinatorial invariants are introduced below and first appeared in [39, Sections 7–9]. An explicit geometric description of these invariants is provided in Corollary 5.11.

Definition 4.9 Let \mathcal{S} be a lattice polyhedral subdivision of a lattice polytope P . Then the **limit mixed h^* -polynomial** of (P, \mathcal{S}) is

$$h^*(P, \mathcal{S}; u, v) := \sum_{F \in \mathcal{S}} v^{\dim F + 1} l^*(F; uv^{-1}) h(\text{lk}_{\mathcal{S}}(F); uv).$$

The **local limit mixed h^* -polynomial** of (P, \mathcal{S}) is

$$l^*(P, \mathcal{S}; u, v) := \sum_{Q \subseteq P} (-1)^{\dim P - \dim Q} h^*(Q, \mathcal{S}|_Q; u, v) g([Q, P]^*; uv).$$

The **refined limit mixed h^* -polynomial** of (P, \mathcal{S}) is

$$h^*(P, \mathcal{S}; u, v, w) = \sum_{Q \subseteq P} w^{\dim Q + 1} l^*(Q, \mathcal{S}|_Q; u, v) g([Q, P]; uvw^2).$$

If \mathcal{S} is the trivial subdivision of P , with cells of \mathcal{S} given by the faces of P , then we write $h^*(P; u, v) = h^*(P, \mathcal{S}; u, v)$ and call the polynomial the **mixed h^* -polynomial**. If P is empty, then $h^*(P, \mathcal{S}; u, v, w) = h^*(P, \mathcal{S}; u, v) = l^*(P, \mathcal{S}; u, v) = 1$.

The following theorem is proved in [39, Theorem 9.2].

Theorem 4.10 Let \mathcal{S} be a lattice polyhedral subdivision of a lattice polytope P . Then the refined limit mixed h^* -polynomial satisfies the following properties:

- (1) The refined limit mixed h^* -polynomial is invariant under the interchange of u and v and satisfies the additional symmetry

$$h^*(P, \mathcal{S}, u, v, w) = h^*(P, \mathcal{S}, u^{-1}, v^{-1}, uvw).$$

- (2) The refined limit mixed h^* -polynomial specializes to the limit mixed h^* -polynomial

$$h^*(P, \mathcal{S}; u, v, 1) = h^*(P, \mathcal{S}; u, v).$$

- (3) The refined limit mixed h^* -polynomial specializes to the mixed h^* -polynomial

$$h^*(P, \mathcal{S}; uw^{-1}, 1, w) = h^*(P; u, w).$$

- (4) The refined limit mixed h^* -polynomial specializes to the h^* -polynomial

$$h^*(P, \mathcal{S}; u, 1, 1) = h^*(P; u).$$

- (5) The degree of $h^*(P, \mathcal{S}; u, v, w)$ as a polynomial in w is at most $\dim P + 1$. Moreover, the coefficient of $w^{\dim P + 1}$ is the local limit mixed h^* -polynomial $l^*(P, \mathcal{S}; u, v)$.

(6) The limit mixed h^* -polynomial can be written in terms of mixed h^* -polynomials,

$$h^*(P, \mathcal{S}; u, v) = \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} (uv - 1)^{\dim P - \dim F} h^*(F; u, v),$$

where ∂P denotes the boundary of P .

In particular, we have the following diagram of invariants

$$\begin{array}{ccccc} h^*(P, \mathcal{S}; u, v, w) & \xrightarrow[v \mapsto 1]{u \mapsto uw^{-1}} & h^*(P; u, w) & & \\ \downarrow w \mapsto 1 & & \downarrow w \mapsto 1 & & \\ h^*(P, \mathcal{S}; u, v) & \xrightarrow{v \mapsto 1} & h^*(P; u) & \xrightarrow{u \mapsto 1} & (\dim P)! \operatorname{vol}(P), \end{array}$$

where $\operatorname{vol}(P)$ is the Euclidean volume of P .

Let Δ_P denote the normal fan to P with all maximal cones removed. The cones γ_Q in Δ_P are in inclusion-reserving correspondence with the positive dimensional faces Q of P . Let Δ'_P denote a simplicial fan refinement of Δ_P which exists by the resolution of singularities algorithm for toric varieties [28, Sec.2.6]. That is, every cone γ' in Δ'_P is generated by precisely $\dim \gamma'$ rays and is contained in a cone of Δ_P . We let $\sigma(\gamma')$ denote the smallest cone in Δ_P containing γ' and set

$$\begin{aligned} \Phi(P, \mathcal{S}, \Delta'_P; u, v, w) \\ = \sum_{\substack{Q \subseteq P \\ \dim Q > 0}} (-1)^{\dim Q} h^*(Q, \mathcal{S}|_Q; u, v, w) \sum_{\substack{\gamma' \in \Delta'_P \\ \sigma(\gamma') = \gamma_Q}} (uvw^2 - 1)^{\dim \gamma_Q - \dim \gamma'}, \end{aligned}$$

and

$$\Lambda(P, \mathcal{S}, \Delta'_P; u, v, w) = \sum_{\gamma' \in \Delta'_P} (uvw^2 - 1)^{\dim P - \dim \gamma'} - \Phi(P, \mathcal{S}, \Delta'_P; u, v, w).$$

We have the following characterization of the refined limit mixed h^* -polynomial is proved in [39, Corollary 9.7].

Corollary 4.11 *The refined limit mixed h^* -polynomial as an invariant of polyhedral subdivisions of lattice polytopes is uniquely characterized by the following properties:*

- (1) *The degree of $h^*(P, \mathcal{S}; u, v, w)$ as a polynomial in w is at most $\dim P + 1$.*
- (2) *The refined limit mixed h^* -polynomial specializes to the limit mixed h^* -polynomial, i.e.,*

$$h^*(P, \mathcal{S}; u, v, 1) = h^*(P, \mathcal{S}; u, v).$$

- (3) *If Δ'_P denotes a simplicial fan refinement of Δ_P then for Λ defined in terms of the refined limit mixed h^* -polynomial as above, we have*

$$\Lambda(P, \mathcal{S}, \Delta'_P; u, v, w) = (uvw^2)^{\dim P + 1} \Lambda(P, \mathcal{S}, \Delta'_P; u^{-1}, v^{-1}, w^{-1}).$$

Similarly, the following characterization of the mixed h^* -polynomial is given in [39, Corollary 9.8]. With the notation above, we set $\Lambda(P, \mathcal{S}, \Delta'_P; u, w) := \Lambda(P, \mathcal{S}, \Delta'_P; uw^{-1}, 1, w)$. Using (3) in Theorem 4.10, we may write this as:

$$\sum_{\gamma' \in \Delta'_P} (uw - 1)^{\dim P - \dim \gamma'} - \sum_{\substack{Q \subseteq P \\ \dim Q > 0}} (-1)^{\dim Q} h^*(Q, \mathcal{S}|_Q; u, w) \\ \times \sum_{\substack{\gamma' \in \Delta'_P \\ \sigma(\gamma') = \gamma_Q}} (uw - 1)^{\dim \gamma_Q - \dim \gamma'}.$$

Corollary 4.12 *The mixed h^* -polynomial as an invariant of lattice polytopes is uniquely characterized by the following properties:*

- (1) *All terms in $h^*(P; u, w)$ have combined degree in u and w at most $\dim P + 1$.*
- (2) *The mixed h^* -polynomial specializes to the h^* -polynomial, i.e.,*

$$h^*(P; u, 1) = h^*(P; u).$$

- (3) *If Δ'_P denotes a simplicial fan refinement of Δ_P then for Λ defined in terms of the mixed h^* -polynomial as above, we have*

$$\Lambda(P, \mathcal{S}, \Delta'_P; u, w) = (uw)^{\dim P + 1} \Lambda(P, \mathcal{S}, \Delta'_P; u^{-1}, w^{-1}).$$

The following example is computed in [39, Example 9.10]:

Example 4.13 If we write

$$h^*(P, \mathcal{S}, u, v, w) = 1 + uvw^2 \cdot \sum_{0 \leq p, q, r \leq \dim P - 1} h_{p, q, r}^*(P, \mathcal{S}) u^p v^q w^r,$$

then we have an explicit description of some of the coefficients of $h^*(P, \mathcal{S}, u, v, w)$. If F is a cell of \mathcal{S} , then let $\sigma(F)$ denote the smallest face of P containing F . Then for $q, r > 0$,

$$h_{0, q, r}^*(P, \mathcal{S}) = \sum_{\substack{F \in \mathcal{S} \\ \dim F = q + 1 \\ \dim \sigma(F) = r + 1}} \#(\text{Int}(F) \cap M), \\ h_{0, 0, r}^*(P, \mathcal{S}) = \sum_{\substack{F \in \mathcal{S} \\ \dim F \leq 1 \\ \dim \sigma(F) = r + 1}} \#(\text{Int}(F) \cap M), \\ \dim P + 1 + h_{0, 0, 0}^*(P, \mathcal{S}) = \sum_{\substack{Q \subseteq P \\ \dim Q \leq 1}} \#(\text{Int}(Q) \cap M).$$

Using Property (1) of Theorem 4.10, when $\dim P = 2$, this gives an explicit description of $h^*(P, \mathcal{S}, u, v, w)$:

$$h^*(P, \mathcal{S}, u, v, w) = 1 + uvw^2 [h_{0, 0, 0}^*(P, \mathcal{S}) + w [(1 + uv)h_{0, 0, 1}^*(P, \mathcal{S}) \\ + (u + v)h_{0, 1, 1}^*(P, \mathcal{S})]].$$

When $\dim P = 3$, we have

$$h^*(P, \mathcal{S}, u, v, w) = 1 + uvw^2 [h_{0, 0, 0}^*(P, \mathcal{S}) + w [(1 + uv)h_{0, 0, 1}^*(P, \mathcal{S}) + (u + v)h_{0, 1, 1}^*(P, \mathcal{S})] \\ + w^2 [(1 + (uv)^2)h_{0, 0, 2}^*(P, \mathcal{S}) + (u + v)(1 + uv)h_{0, 1, 2}^*(P, \mathcal{S}) \\ + (u^2 + v^2)h_{0, 2, 2}^*(P, \mathcal{S}) + uvh_{1, 1, 2}^*(P, \mathcal{S})]],$$

where each term has an explicit description above except $h_{1, 1, 2}^*(P, \mathcal{S})$. By (4) of Theorem 4.10, $h^*(P, \mathcal{S}, 1, 1, 1) = h^*(P, \mathcal{S}, 1) = 6 \text{ vol}(P)$, and this determines $h_{1, 1, 2}^*(P, \mathcal{S})$ and hence $h^*(P, \mathcal{S}, u, v, w)$.

5 Refined limit mixed Hodge numbers of hypersurfaces

The goal of this section is to present a proof of Theorem 1.5 giving a combinatorial formula for the refined limit Hodge–Deligne polynomial of a schön hypersurface in $(\mathbb{K}^*)^n$ which is interpreted as a family of hypersurfaces. We first reprove a theorem of Danilov–Khovanskii for the χ_Y -characteristic of a complex hypersurface in terms of the h^* -polynomial of its Newton polytope. Then we give combinatorial formulas of the following progressively finer cohomological invariants: the Hodge–Deligne polynomial of a generic fiber; the limit Hodge–Deligne polynomial, the limit Hodge–Deligne polynomial of a smooth compactification of the family of hypersurfaces, and then the refined limit Hodge–Deligne polynomial. We will make use of the fact that the cohomology of a hypersurface is tightly constrained by Poincaré duality and the weak Lefschetz theorem.

5.1 Tropical geometry for hypersurfaces

Let $X^\circ = \{\sum_{u \in M} \alpha_u x^u = 0\} \subset T \cong (\mathbb{K}^*)^n$ be a schön hypersurface. The **Newton polytope** P of X° is the convex hull of $\{u \in M \mid \alpha_u \neq 0\}$. Note that P may be viewed as a full-dimensional lattice polytope in the translation M of the saturation of its integer affine span in \mathbb{Z}^n to the origin, and $X^\circ \cong X' \times (\mathbb{K}^*)^k$, for some k , where $X' \subseteq \text{Spec } \mathbb{K}[M]$ is a schön hypersurface with Newton polytope P . Hence we may and will assume that $\dim P = n$.

Tropical geometry of hypersurfaces reduces to the study of Newton polytopes and polyhedral subdivisions [29, 51]. Recall that the field \mathbb{K} has a natural valuation by considering the vanishing order of a function on \mathbb{D}^* at the origin. With the notation above, the function $P \cap \mathbb{Z}^n \rightarrow \mathbb{Z}, u \mapsto \text{ord}(\alpha_u)$ induces a regular, lattice subdivision \mathcal{S} of P . Explicitly, the cells of \mathcal{S} are the projections of the bounded faces of the convex hull of $\text{UH} = \{(u, \lambda) \mid \alpha_u \neq 0, \lambda \geq \text{ord}(\alpha_u)\}$ in $\mathbb{R}^n \times \mathbb{R}$, and the bounded faces of UH are the graph of a function $\omega : P \rightarrow \mathbb{R}$. Restricting to $P \cap \mathbb{Z}^n$, we get a height function. There is a dual complex associated with the height function that generalizes the normal fan. The cells of this complex are in inclusion-reversing bijective correspondence with the cells of \mathcal{S} . See [32, 9.11] for details.

Remark 5.1 Since the initial degeneration $\text{in}_w X^\circ$ of a hypersurface is given by the corresponding initial form of its defining polynomial, for a generic choice of coefficients (in a certain analytic topology), a hypersurface with a given height function is schön, i.e., all initial degenerations are smooth. Hence every pair (P, \mathcal{S}) , where \mathcal{S} is a regular, lattice polyhedral subdivision of a lattice polytope P arises from the construction above for some schön hypersurface. See [33, Section 8.1] for a more detailed discussion of genericity and schönness.

The tropicalization $\text{Trop}(X^\circ)$ is supported on the non-maximal-dimensional skeleton of the dual complex [51, Section 3] to \mathcal{S} . The restriction of the dual complex to $\text{Trop}(X^\circ)$ gives a polyhedral structure Σ . With the notation of Sect. 4, the recession fan $\Delta = \Delta_P$ of Σ is the normal fan of P with the maximal cones removed. Recall from Sect. 2 that we may define a toric scheme $\mathbb{P}(\Sigma)_\mathcal{O}$ over \mathcal{O} from Σ with generic fiber equal to the toric variety $\mathbb{P}(\Delta)_\mathbb{K}$. Let \mathcal{X} denote the closure of X° in $\mathbb{P}(\Sigma)_\mathcal{O}$, and let X_Δ and X_0 denote the generic fiber and central fiber of \mathcal{X} , respectively. Then we can write the stratifications of X_Δ and X_0 in dual language with respect to the Newton polytope and subdivision as the following:

$$X_\Delta = \bigcup_{\substack{Q \subseteq P \\ \dim Q > 0}} X_Q^\circ, \quad X_0 = \bigcup_{\substack{F \in S \\ \dim F > 0}} X_F^\circ$$

The fixed nonzero fiber X_{gen}° is a schön hypersurface with Newton polytope P in its corresponding complex torus, which we denote as T_{gen} . For every cell F of S with $\dim F > 0$, the corresponding complex variety X_F° is a complex schön hypersurface with Newton polytope F , and, if w lies in the relative interior of the cell in Σ corresponding to F , then

$$\text{in}_w X^\circ \cong X_F^\circ \times (\mathbb{C}^*)^{\dim P - \dim F}.$$

When $\dim F = 0$, $\text{in}_w X^\circ = X_F^\circ = \emptyset$, and the corresponding motivic invariants are zero. We conclude that Theorem 1.2 translates into the following corollary.

Corollary 5.2 *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, S) and $\dim P = n$. Then the motivic nearby fiber of X° is given by*

$$\psi_{X^\circ} = \sum_{\substack{F \in S \\ F \not\subseteq \partial P}} [X_F^\circ](1 - \mathbb{L})^{\dim P - \dim F},$$

where ∂P denotes the boundary of P , $\mathbb{L} := [\mathbb{A}^1] \in K_0(\text{Var}_{\mathbb{C}})$, and X_F° is a complex schön hypersurface with Newton polytope F .

5.2 The χ_y -characteristic of a complex hypersurface

We apply Corollary 5.2 to give a new proof of a formula of Danilov–Khovanskii [20, Section 4] for the χ_y -characteristic of schön hypersurfaces in $(\mathbb{C}^*)^n$.

Remark 5.3 The fact that the Hodge–Deligne polynomial of a schön hypersurface of a complex torus is determined by its Newton polytope can be seen directly. One considers the closure V of a schön hypersurface V° given by a Laurent polynomial with Newton polytope P in a toric resolution of the complex toric variety determined by P . It is a smooth variety. Since Hodge numbers are locally constant through families of smooth proper varieties, the Hodge–Deligne polynomial of V is independent of the choice of polynomial. The result can then be deduced from the motivic nature of the Hodge–Deligne polynomial.

Let V° be a schön hypersurface of a complex torus given by a polynomial with Newton polytope P . We may suppose $\dim P = \dim V^\circ + 1$. Recall from Remark 5.1 and Remark 5.3 that the Hodge–Deligne polynomial of V° only depends on P and that given any P , there exists a corresponding schön hypersurface V° . Hence we may define

$$E(V_P^\circ; u, v) := E(V^\circ; u, v).$$

If P is empty, then we let V_P° be the empty set. To identify the χ_y -characteristic, we build a valuation out of it (see Definition 4.6).

Lemma 5.4 *The map*

$$\begin{aligned} \mathcal{P}_{\mathbb{Z}^n} &\rightarrow \mathbb{Z}[[u]] \\ P &\mapsto \frac{E(V_P^\circ; u, 1)}{(u - 1)^{\dim P + 1}} \end{aligned}$$

is a valuation on the set $\mathcal{P}_{\mathbb{Z}^n}$ of lattice polytopes in \mathbb{Z}^n .

Proof Properties (2) and (3) in Definition 4.6 are clearly satisfied so we must show Property (1). Let \mathcal{S} be a regular lattice polyhedral subdivision of P . By Remark 5.1, there exists a schön hypersurface $X^\circ \subset (\mathbb{K}^*)^{\dim P}$ with corresponding Newton polytope P and polyhedral subdivision \mathcal{S} . This hypersurface satisfies $E(X_{\text{gen}}^\circ; u, 1) = E(V_P^\circ; u, 1)$. By Corollary 5.2 and (9), we obtain

$$E(V_P^\circ; u, 1) = \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} E(V_F^\circ; u, 1)(1 - u)^{\dim P - \dim F}$$

If we divide by $(u - 1)^{\dim P + 1}$, we get

$$\frac{E(V_P^\circ; u, 1)}{(u - 1)^{\dim P + 1}} = \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} (-1)^{\dim P - \dim F} \frac{E(V_F^\circ; u, 1)}{(u - 1)^{\dim F + 1}}.$$

□

With the notation of Sect. 4, we obtain a new proof of Danilov and Khovanskii's theorem.

Theorem 5.5 [20, Sec. 4] *Let P be a non-empty lattice polytope and let V_P° be a complex schön hypersurface with Newton polytope P . Then we have the following formula for the χ_y -characteristic of V_P° :*

$$uE(V_P^\circ; u, 1) = (u - 1)^{\dim P} + (-1)^{\dim P + 1} h^*(P; u),$$

where $h^*(P; u)$ is the h^* -polynomial of P .

Proof We continue with the notation of Lemma 5.4. By dividing both sides of the equation by $(u - 1)^{\dim P + 1}$, it suffices to establish the following:

$$\frac{uE(V_P^\circ; u, 1)}{(u - 1)^{\dim P + 1}} = \frac{f_0(P)}{u - 1} + \sum_{m \geq 0} f_P(m) u^m$$

By Lemma 5.4 and Example 4.8, both sides are valuations. By Lemma 4.7, we need only check the case of unimodular simplices Δ_l . In that case, a straightforward computation [10, Sec. 2.3] gives $h^*(\Delta_l) = 1$ and we need to check that

$$uE(V_{\Delta_l}^\circ; u, 1) = (u - 1)^l + (-1)^{l+1}.$$

We prove this by induction. For $l = 0$, both sides of the equation are 0. For $l \geq 1$, $V_{\Delta_l}^\circ$ is the intersection of a generic hyperplane in \mathbb{P}^l with $(\mathbb{C}^*)^l$. This is isomorphic to the complement of $l + 1$ generic hyperplanes in \mathbb{P}^{l-1} . By treating l of these hyperplanes as coordinate hyperplanes and the last one as some generic hyperplane, we get the motivic relation $[V_{\Delta_l}] = [(\mathbb{C}^*)^{l-1}] - [V_{\Delta_{l-1}}]$. Because $V^\circ \mapsto E(V^\circ; u, 1)$ is motivic and $E((\mathbb{C}^*)^{l-1}; u, 1) = (u - 1)^{l-1}$, we have

$$\begin{aligned} uE(V_{\Delta_l}^\circ; u, 1) &= u(u - 1)^{l-1} - (u - 1)^{l-1} - (-1)^l \\ &= (u - 1)^l + (-1)^{l+1}. \end{aligned}$$

□

By specializing the above theorem to $u = 1$ and using the fact that $h^*(P; 1) = (\dim P)! \text{vol}(P)$ where $\text{vol}(P)$ is the Euclidean volume of P , we get the following well-known result of Kouchnirenko [40]:

Corollary 5.6 *Let P be a non-empty lattice polytope and let V_P° be a schön hypersurface with Newton polytope P . Then we have the following formula for the topological Euler characteristic of V_P° :*

$$e(V_P^\circ) = (-1)^{\dim P + 1} (\dim P)! \operatorname{vol}(P).$$

5.3 A Danilov–Khovanskii type algorithm

In [20], Danilov and Khovanskii use their formula for the χ_y -characteristic in Theorem 5.5 in connection with the weak Lefschetz theorem and Poincaré duality to give an algorithm to compute the Hodge–Deligne polynomial of a complex schön hypersurface. We use an analogous approach to providing an algorithm to compute the refined limit Hodge–Deligne polynomial of a schön hypersurface from the limit Hodge–Deligne polynomial. We continue with the notation from earlier in this section.

We consider the cohomology with compact supports of the complex variety $X_{\text{gen}}^\circ \subseteq T_{\text{gen}}$, and set $n = \dim T_{\text{gen}}$. The following weak Lefschetz result implies that the only interesting cohomology is in middle dimension.

Proposition 5.7 [20, Proposition 3.9] *The Gysin map $H_c^k(X_{\text{gen}}^\circ) \rightarrow H_c^{k+2}(T_{\text{gen}})$ is an isomorphism for $k > n - 1$, and a surjection for $k = n - 1$. Since X_{gen}° is affine, $H_c^k(X_{\text{gen}}^\circ) = 0$ for $k < n - 1$.*

Indeed, the Gysin map above is a morphism of mixed Hodge structures of type $(1, 1)$, and hence the (usual) mixed Hodge structure on $H_c^k(X_{\text{gen}}^\circ)$ is known for $k \neq n - 1$ by Example 3.1. Following [7], we define the **primitive cohomology** of X_{gen}° to be

$$H_{c,\text{prim}}^{n-1} X_{\text{gen}}^\circ := \ker \left[H_c^{n-1} X_{\text{gen}}^\circ \rightarrow H_c^{n+1} T_{\text{gen}} \right],$$

with the induced mixed Hodge structure. Since the Gysin map varies naturally in families over \mathbb{D}^* , it commutes with the monodromy operator, and so by Example 3.11, the corresponding nilpotent operator N preserves the primitive cohomology of X_{gen}° .

It follows that the refined limit Hodge–Deligne polynomial $E(X_\infty^\circ; u, v, w)$ determines and is determined by the refined limit Hodge numbers of the primitive cohomology of X_∞° . In particular, we have the following lemma:

Lemma 5.8 *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, S) . Then, as a polynomial in w , $uvw^2 E(X_\infty^\circ; u, v, w)$ has the same coefficient as $(uvw^2 - 1)^{\dim P + 1}$ in all degrees strictly greater than $\dim P + 1$.*

Proof Since X_{gen}° is a smooth complex variety, the graded pieces of the Deligne weight filtration $Gr_r^W H_c^m(X_{\text{gen}}^\circ)$ are zero for $r > m$ by, e.g., [50, Thm 5.39]. In particular, the contributions from the primitive cohomology of X_{gen}° to $E(X_\infty^\circ; u, v, w)$ all have degree at most $\dim P - 1$ in w . The result then follows from the above discussion and Example 3.11. \square

The above lemma may be viewed as a generalization of the corresponding statement for the Hodge–Deligne polynomial, due to Danilov and Khovanskii, which follows by the exact same argument as above.

Lemma 5.9 [20, Sec. 3.11] *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope P . Then the coefficient of $u^p w^q$ in $uwE(X_{\text{gen}}^\circ; u, w)$ equals the coefficient of $u^p w^q$ in $(uw - 1)^{\dim P + 1}$ for $p + q > \dim P + 1$.*

We next explain the use of Poincaré duality. Recall that the recession fan Δ_P is the normal fan to P with all maximal cones removed, with cones γ_Q in inclusion-reserving correspondence with the positive dimensional faces Q of P . As in Sect. 4, let Δ'_P denote a simplicial fan refinement of Δ_P , and let $\sigma(\gamma')$ denote the smallest cone in Δ_P containing a cone γ' in Δ'_P . Then we have an induced proper, birational map of toric varieties over \mathbb{K} , $\pi : \mathbb{P}(\Delta'_P)_{\mathbb{K}} \rightarrow \mathbb{P}(\Delta_P)_{\mathbb{K}}$, which, by standard toric geometry, is locally a projection in the sense that if $\mathbb{P}(\Delta'_P)_{\mathbb{K}} = \bigcup_{\gamma' \in \Delta'_P} U_{\gamma'}$ and $\mathbb{P}(\Delta_P)_{\mathbb{K}} = \bigcup_{Q \subseteq P, \dim Q > 0} U_{\gamma_Q}$ are unions of the toric varieties into torus orbits, then $\pi|_{U_{\gamma'}}$ is given by

$$\pi|_{U_{\gamma'}} : U_{\gamma'} \cong U_{\sigma(\gamma')} \times (\mathbb{K}^*)^{\dim \sigma(\gamma') - \dim \gamma'} \rightarrow U_{\sigma(\gamma')}.$$

Let X'_P and X_P denote the closure of X° in the toric varieties $\mathbb{P}(\Delta'_P)_{\mathbb{K}}$ and $\mathbb{P}(\Delta_P)_{\mathbb{K}}$, respectively. Then X'_P is proper and has at worst orbifold singularities. The possibly singular variety X_P has a stratification into schön subvarieties

$$X_P = \bigcup_{\substack{Q \subseteq P \\ \dim Q > 0}} X_Q^\circ,$$

where $X^\circ = X_P^\circ$, and X_Q° corresponds to the pair $(Q, \mathcal{S}|_Q)$. We conclude that

$$E(X'_{P,\infty}; u, v, w) = \sum_{\substack{Q \subseteq P \\ \dim Q > 0}} E(X_{Q,\infty}^\circ; u, v, w) \sum_{\substack{\gamma' \in \Delta'_P \\ \sigma(\gamma') = \gamma_Q}} (uvw^2 - 1)^{\dim \gamma_Q - \dim \gamma'}. \quad (11)$$

Since X'_P is proper and has at worst orbifold singularities, Poincaré duality [50, Prop 6.19] implies that

$$E(X'_{P,\infty}; u, v, w) = (uvw^2)^{\dim P - 1} E(X'_{P,\infty}; u^{-1}, v^{-1}, w^{-1}). \quad (12)$$

We conclude that we have the following algorithm to determine $E(X_\infty^\circ; u, v, w)$ from $E(X_\infty^\circ; u, v, 1) = E(X_\infty^\circ; u, v)$, using induction on dimension. Consider $E(X_\infty^\circ; u, v, w)$ as a polynomial in w . Firstly, Lemma 5.8 implies that we know $E(X_\infty^\circ; u, v, w)$ in all degrees strictly greater than $\dim P - 1$. Secondly, by induction on dimension and (11), we know $E(X'_{P,\infty}; u, v, w)$ in all degrees strictly greater than $\dim P - 1$, and by (12), we know $E(X'_{P,\infty}; u, v, w)$ and hence $E(X_\infty^\circ; u, v, w)$ in all degrees strictly less than $\dim P - 1$. Finally, $E(X_\infty^\circ; u, v, 1)$ now determines $E(X_\infty^\circ; u, v, w)$ in degree $\dim P - 1$.

Remark 5.10 The same argument gives the Danilov and Khovanskii algorithm to determine the Hodge–Deligne polynomial $E(X_\infty^\circ; uw^{-1}, 1, w) = E(X_{\text{gen}}^\circ; u, w)$ from the χ_y -characteristic $E(X_\infty^\circ; u, 1, 1) = E(X_{\text{gen}}^\circ; u, 1)$. Explicitly, Lemma 5.9 implies that we know the coefficient of $u^p w^q$ in $E(X_{\text{gen}}^\circ; u, w)$ for $p + q > \dim P - 1$. Secondly, by induction on dimension and (11), we know the coefficient of $u^p w^q$ in $E(X'_{P,\text{gen}}; u, w)$ for $p + q > \dim P$, and by (12), we know the coefficient of $u^p w^q$ in $E(X'_{P,\text{gen}}; u, w)$ and hence $E(X_{\text{gen}}^\circ; u, w)$ in all degrees strictly less than $\dim P - 1$. Finally, $E(X_{\text{gen}}^\circ; u, 1)$ now determines the coefficient of $u^p w^q$ in $E(X_{\text{gen}}^\circ; u, w)$ when $p + q = \dim P - 1$.

5.4 A formula for the refined limit Hodge–Deligne polynomial

We now complete the proof of Theorem 1.5 and deduce an explicit description of the refined limit mixed Hodge numbers of a schön hypersurface. We will see that the proof

reduces to some combinatorial results which are proved in [39]. We also state some immediate consequences of the theorem.

Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, \mathcal{S}) and $\dim P = n$. We will work our way through the diagram

$$\begin{array}{ccc} E(X_\infty^\circ; u, v, w) & \longrightarrow & E(X_{\text{gen}}^\circ; u, w) \\ \downarrow & & \downarrow \\ E(X_\infty^\circ; u, v) & \longrightarrow & E(X_{\text{gen}}^\circ; u, 1) \longrightarrow e(X_{\text{gen}}^\circ), \end{array}$$

In Sect. 5.2, we proved the formula

$$uE(X_{\text{gen}}^\circ; u, 1) = (u - 1)^{\dim P} + (-1)^{\dim P+1} h^*(P; u),$$

where $h^*(P; u)$ is the h^* -polynomial of P . We claim that

$$uwE(X_{\text{gen}}^\circ; u, w) = (uw - 1)^{\dim P} + (-1)^{\dim P+1} h^*(P; u, w).$$

This is the Borisov–Mavlyutov formula for the Hodge–Deligne polynomial [15]. We will prove this formula using the method of [56]. Indeed, we only need to verify that the proposed formula satisfies the algorithm of Remark 5.10. Recall that the algorithm consists of three parts: weak Lefschetz, specialization, and Poincaré duality. That the proposed formula satisfies the weak Lefschetz property (Lemma 5.9) follows from (1) in Corollary 4.12. The fact that the proposed formula specializes to the formula for $E(X_{\text{gen}}^\circ; u, 1)$ when setting $w = 1$ follows from (2) in Corollary 4.12. Finally, that the proposed formula satisfies the Poincaré duality property follows by substitution into (11) (after specializing $u \mapsto uw^{-1}$, $v \mapsto 1$) and (3) in Corollary 4.12.

To determine the limit Hodge–Deligne polynomial, we note that Corollary 5.2 specializes under application of the Hodge–Deligne map to the formula

$$E(X_\infty^\circ; u, v) = \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} E(X_F^\circ; u, v) (1 - uv)^{\dim P - \dim F}.$$

Substituting the Borisov–Mavlyutov formula for the Hodge–Deligne polynomial yields, using Lemma 2.1,

$$\begin{aligned} uvE(X_\infty^\circ; u, v) &= \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} [(uv - 1)^{\dim F} + (-1)^{\dim F+1} h_F^*(u, v)] (1 - uv)^{\dim P - \dim F} \\ &= (uv - 1)^{\dim P} + (-1)^{\dim P+1} \sum_{\substack{F \in \mathcal{S} \\ F \not\subseteq \partial P}} h_F^*(u, v) (uv - 1)^{\dim P - \dim F}. \end{aligned}$$

Now (6) in Theorem 4.10 gives our desired formula

$$uvE(X_\infty^\circ; u, v) = (uv - 1)^{\dim P} + (-1)^{\dim P+1} h^*(P, \mathcal{S}; u, v).$$

Finally, we want to prove

$$uvw^2E(X_\infty^\circ; u, v, w) = (uvw^2 - 1)^{\dim P} + (-1)^{\dim P+1} h^*(P, \mathcal{S}; u, v, w).$$

It remains to show that the proposed formula satisfies the three parts of the algorithm in Sect. 5.3. That the proposed formula satisfies the weak Lefschetz property (Lemma 5.8) follows from (1) in Corollary 4.11. The fact that the proposed formula specializes to the

above formula for $E(X_\infty^\circ; u, v)$ when setting $w = 1$ follows from (2) in Corollary 4.11. That the proposed formula satisfies the Poincaré duality property follows by substitution into (11) and then applying (3) in Corollary 4.11.

Using our description of the cohomology of X_{gen}° in Sects. 5.3 and 5.1 together with the above formula, we immediately deduce the following corollary. The second two statements below follow from (2) and (5) in Theorem 4.10, respectively. We refer the reader to Example 4.13 and Theorem 4.10 for explicit combinatorial descriptions of the invariants below in the cases when $n = 2, 3$.

Corollary 5.11 *Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, \mathcal{S}) and $\dim P = n$. Then the refined limit mixed Hodge numbers associated with the primitive cohomology of X° are given by*

$$h^*(P, \mathcal{S}; u, v, w) = 1 + uvw^2 \sum_{p,q,r} h^{p,q,r}(H_{\text{prim},c}^{\dim P-1}(X_\infty^\circ)) u^p v^q w^r.$$

In particular, the corresponding limit mixed Hodge numbers are given by

$$h^*(P, \mathcal{S}; u, v) = 1 + uv \sum_{p,q} h^{p,q}(H_{\text{prim},c}^{\dim P-1}(X_\infty^\circ)) u^p v^q.$$

Moreover, the limit mixed Hodge numbers of $Gr_{\dim P-1}^W H_{\text{prim},c}^{\dim P-1}(X_\infty^\circ)$ are given by

$$l^*(P, \mathcal{S}; u, v) = uv \sum_{p,q} h^{p,q}(Gr_{\dim P-1}^W H_{\text{prim},c}^{\dim P-1}(X_\infty^\circ)) u^p v^q.$$

As in Corollary 2.4, the motivic nature of the invariants above means that we obtain formulas for invariants of partial compactifications of schön hypersurfaces. We now state this explicitly for possible future reference.

Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, \mathcal{S}) and $\dim P = n$. With the notation of Sect. 5.3, the recession fan Δ_P has cones γ_Q in inclusion-reserving correspondence with the positive dimensional faces Q of P . Let $\tilde{\Delta}'_P$ denote a fan refinement (not necessarily simplicial) of a subfan $\tilde{\Delta}_P$ of Δ_P . We let $\sigma(\gamma')$ denote the smallest cone in $\tilde{\Delta}_P$ containing γ' . Let \tilde{X}'_P denote the closure of X° in the toric variety $\mathbb{P}(\tilde{\Delta}'_P)_{\mathbb{K}}$ over \mathbb{K} . Let \tilde{X}'_P and \tilde{X}_P denote the closure of X° in the toric varieties $\mathbb{P}(\tilde{\Delta}'_P)_{\mathbb{K}}$ and $\mathbb{P}(\Delta_P)_{\mathbb{K}}$, respectively. Then \tilde{X}'_P and \tilde{X}_P have toroidal singularities, and X_P has a stratification into schön subvarieties

$$X_P = \bigcup_{\substack{Q \subseteq P \\ \dim Q > 0}} X_Q^\circ,$$

where X_Q° corresponds to the pair $(Q, \mathcal{S}|_Q)$. The arguments of Sect. 5.3 imply the following expressions for the refined limit Hodge–Deligne polynomial and motivic nearby fiber of \tilde{X}'_P , respectively.

$$\begin{aligned} E(\tilde{X}'_P; u, v, w) &= \sum_{\substack{Q \subseteq P \\ \dim Q > 0}} E(X_Q^\circ; u, v, w) \sum_{\substack{\gamma' \in \tilde{\Delta}'_P \\ \sigma(\gamma') = \gamma_Q}} (uvw^2 - 1)^{\dim \gamma_Q - \dim \gamma'}, \\ \psi_{\tilde{X}'_P} &= \sum_{\substack{Q \subseteq P \\ \dim Q > 0}} \psi_{X_Q^\circ} \sum_{\substack{\gamma' \in \tilde{\Delta}'_P \\ \sigma(\gamma') = \gamma_Q}} (\mathbb{L} - 1)^{\dim \gamma_Q - \dim \gamma'}. \end{aligned}$$

Finally, combinatorial expressions for $E(X_{Q,\infty}^\circ; u, v, w)$ and $\psi_{X_Q^\circ}$ are given in Theorem 1.5 and Corollary 5.2, respectively. Note that we may allow the summations above to run over all non-empty faces Q of P since $X_Q^\circ = \emptyset$ when $\dim Q = 0$.

Remark 5.12 In the case when $\Delta_P = \tilde{\Delta}'_P = \tilde{\Delta}_P$ above, $X_P := \tilde{X}'_P$ is the closure of X° in the toric variety $\mathbb{P}(\Delta_P)_\mathbb{K}$ over \mathbb{K} . In this case, X_P is proper but singular in general. Interestingly, we have a purely combinatorial expression for $E(X_{P,\infty}; u, v, w)$ above, although one cannot hope to obtain a combinatorial expression for the Betti numbers of X_P , nevermind the refined limit mixed Hodge numbers of X_P since the Betti numbers of the ambient toric variety are not combinatorial [25, 44]. We obtain the following expression for the motivic nearby fiber

$$\psi_{X_P} = \sum_{\emptyset \neq Q \subseteq P} \psi_{X_Q^\circ} = \sum_{\emptyset \neq F \in \mathcal{S}} [X_F^\circ](1 - \mathbb{L})^{\dim \sigma(F) - \dim F},$$

where $\sigma(F)$ denotes the smallest face of P containing F . If we further assume that Δ_P is smooth, then the above expression for the motivic nearby fiber appeared in [38, Section 6].

Finally, we present the following application of Theorem 1.5.

Example 5.13 Let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, \mathcal{S}) . Let X denote the closure of X° in the projective toric variety over \mathbb{K} corresponding to the normal fan of P . We assume that P is **reflexive** in the sense of Batyrev [5, Section 4.1]. That is, we assume that P contains the origin in its relative interior, and the associated dual polytope P^* is also a lattice polytope. In this case, there is an inclusion-reversing correspondence between faces Q of P and faces Q^* of P^* . Moreover, X_{gen} is a projective Calabi–Yau variety with at worst canonical singularities. Similarly, let X^* denote a family of projective Calabi–Yau varieties corresponding to the pair (P^*, \mathcal{S}^*) , for some polyhedral subdivision \mathcal{S}^* of P^* .

Batyrev introduced the notion of a **stringy invariant** $E_{st}(V; u, w)$ of a complex variety V with at worst log-terminal singularities in [6], such that if V admits a crepant resolution V' then $E_{st}(V; u, w) = E(V'; u, w)$. In [15, Theorem 7.2], Borisov and Mavlyutov proved a result equivalent to the following formula for the projective complex variety $V = X_{\text{gen}}$:

$$uvw^2 E_{st}(X_{\text{gen}}; u, w) = \sum_{Q \subseteq P} (-w)^{\dim Q + 1} l^*(Q, \mathcal{S}|_Q; uw^{-1}) l^*(Q^*; uw).$$

This formula greatly simplified an earlier formula of Batyrev and Borisov [8, Theorem 4.14]. Moreover, it follows immediately that $E_{st}(X_{\text{gen}}; u, w) = u^{\dim P - 1} E_{st}(X_{\text{gen}}^*; u^{-1}, w)$, which is precisely Batyrev and Borisov’s mirror symmetry construction for Calabi–Yau hypersurfaces [8, Theorem 4.14].

One may extend the definition of stringy invariants to varieties over \mathbb{K} and define a polynomial $E_{st}(X; u, v, w)$, which agrees with the refined limit Hodge–Deligne polynomial of a crepant resolution of X over \mathbb{K} . Using the methods of [15], together with Theorem 1.5, yields the formula

$$uvw^2 E_{st}(X; u, v, w) = \sum_{Q \subseteq P} (-w)^{\dim Q + 1} l^*(Q, \mathcal{S}|_Q; u, v) l^*(Q^*; uvw^2).$$

We observe that there is no direct relation between $E_{st}(X; u, v, w)$ and $E_{st}(X^*; u, v, w)$, except in the case when both X and X^* are either trivial degenerations or maximally

degenerate, in the sense that all nonzero limit mixed Hodge numbers are of type (p, p) , in which case one recovers a statement equivalent to the Batyrev–Borisov result above.

6 Intersection cohomology of schön subvarieties

In this section, we give a sum-over strata formula for the refined limit Hodge–Deligne polynomial of the intersection cohomology of the closure of a schön subvariety in certain projective toric varieties over a punctured curve C . This formula is analogous to a special case of the motivic formula obeyed by the usual Hodge–Deligne polynomial. It differs in that it only works for stratifications induced by the ambient toric variety and that it requires a weighting of terms by the g -polynomial to account for singularities along strata. By considering the case of schön hypersurfaces, we will give an alternative proof of Theorem 1.5 for families of schön hypersurfaces over a punctured curve.

We will let $\mathbb{K} = \mathbb{C}(t)$ instead of the function field of germs of analytic functions on a punctured disk. We view varieties defined over \mathbb{K} as algebraic families of varieties over a curve C that has a distinguished puncture 0. All monodromy will be computed around this puncture.

6.1 Sum-over-strata formulas in intersection cohomology

In the proof [8] of their formula for the cohomology of a schön hypersurface of a toric variety, Batyrev and Borisov observe that the intersection cohomology of schön hypersurfaces in the projective toric variety associated with their Newton polytope obeys a sum-over-strata formula analogous to that of the cohomology of projective toric varieties. Cappell, Maxim, and Shaneson [18] who study what they call the ‘stratified multiplicative property of intersection cohomology’ prove a natural generalization of that observation. They study an intersection cohomology Euler characteristic (such as topological Euler characteristic, χ_Y -characteristic, or Hodge–Deligne polynomial), extend its definition to open strata and study how it behaves under a stratified fibration $f : X \rightarrow Y$. One can generalize these results to give a sum-over-strata formula for the refined limit Hodge–Deligne polynomial of a family of schön subvarieties over a punctured disc.

We first establish our framework. We will use middle-perversity throughout. All stratification will be complex algebraic stratifications. The compactly supported intersection cohomology of a quasi-projective variety over \mathbb{C} has a mixed Hodge structure [53] and therefore one can define a Hodge–Deligne polynomial. For a quasi-projective variety X° over a curve C , the intersection cohomology with compact supports of the family forms a mixed Hodge module by the work of Saito [52]. In an arbitrarily small disk around the puncture, we can suppose that this is an admissible variation of mixed Hodge structures. Therefore, we have a Hodge, weight, and monodromy weight filtration on the compactly supported intersection cohomology of a generic fiber, and as in Sect. 3.2, we can form the refined limit Hodge–Deligne polynomial and refined limit mixed Hodge numbers. The advantage of using intersection cohomology is that projective varieties carry a *pure* Hodge structure and that toric strata can be treated as if they are smoothly embedded once one uses a combinatorial correction term coming from the g -polynomial. Let X° be a schön subvariety of $(\mathbb{K}^*)^n$. Let Δ_P be the normal fan of a lattice polytope P such that the recession fan of $\text{Trop}(X^\circ)$ is supported on Δ_P , i.e., the support of the recession fan is a union of cones in Δ_P . Such a polytope always exists by arguments using the Hilbert scheme [60]. In this case, we say $\mathbb{P}(\Delta_P)$ is **adapted** to X° . If X° is a schön hypersurface, it

suffices to take P to be the Newton polytope of X° . We will let X be the closure of X° in $\mathbb{P}(\Delta_P)_\mathbb{K}$. We say that X is a **schön**, projective variety. Note that for $X^\circ = (\mathbb{K}^*)^n$, we have $X = \mathbb{P}(\Delta_P)_\mathbb{K}$. We may also study the case where X° is the schön subvariety of $(\mathbb{C}^*)^n$. In this case, we say $\mathbb{P}(\Delta_P)_\mathbb{K}$ is adapted if it is adapted to $X^\circ \times_{\mathbb{C}} \mathbb{K}$.

We begin with the analog of the sum-over-strata formula analogous to the motivic formula for compactly supported cohomology. Please note that our convention for the g -polynomial differs from that of [8].

Theorem 6.1 *Let $X \subseteq \mathbb{P}(\Delta_P)_\mathbb{K}$ be the closure of a schön subvariety in an adapted projective toric variety. The refined limit Hodge–Deligne polynomial obeys*

$$E_{\text{int}}(X_\infty; u, v, w) = \sum_{\sigma \in \Delta_P} E((X_\sigma^\circ)_\infty; u, v, w) g([0, \sigma]; uvw^2).$$

Proof The analogous formula for the Hodge–Deligne polynomial for varieties over \mathbb{C} is deduced from the decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [9] in [8, Corollary 3.17]. There it is stated for hypersurfaces, but the arguments also hold for the closure of schön subvarieties in adapted projective toric varieties. Work in a similar direction has been done by Cappell et al. [18] who study a ‘stratified multiplicative property’ which is proved for the Hodge–Deligne polynomial in intersection cohomology. Again, the result is an application of the decomposition theorem. One can derive the sum-over-strata formula from the stratified multiplicative property as follows: one takes a projective toric resolution of singularities $f : \mathbb{P}(\Delta_{\tilde{P}}) \rightarrow \mathbb{P}(\Delta_P)$; there is an induced resolution of singularities of the closures of the schön subvariety, $f : \tilde{X} \rightarrow X$ to which one applies the stratified multiplicative property; and one then deduces the sum-over-strata formula from the analogous formula on \tilde{X} where the intersection cohomology Hodge–Deligne polynomial reduces to the usual Hodge–Deligne polynomial which is known to be motivic.

To justify the formula for the refined limit Hodge–Deligne polynomial, we use the approach of [18]. Their results are stated for stratifications with simply connected strata, but they note that they only need the property that the local systems involved in the decomposition theorem have trivial monodromy along strata. This property is verified for schön hypersurfaces in [8, Cor 3.17]. The same proof holds for schön subvarieties. To obtain the result for the limit mixed Hodge structure coming from a family, it suffices to show that the isomorphism in the decomposition theorem respects the monodromy weight filtration.

Let X_t be a family of schön subvarieties of a projective toric variety $\mathbb{P}(\Delta_P)_C$ over a (possibly non-proper) curve C . Here, we will be concerned with the monodromy around 0. Write $p : X \rightarrow C$. Take a toric resolution of singularities $f : \mathbb{P}(\Delta_{\tilde{P}})_C \rightarrow \mathbb{P}(\Delta_P)_C$, and let \tilde{X} be the closure of X° in $\mathbb{P}(\Delta_{\tilde{P}})$. By applying the decomposition theorem to the resolution of singularities $f : \tilde{X} \rightarrow X$, we have a non-canonical isomorphism,

$$(p \circ f)_* \mathbb{Q}_{\tilde{X}}[n+1] \cong \bigoplus_i \bigoplus_{l=0}^{\dim Y} p_* \left(IC_{\tilde{S}_l}(L_{i,l})[-i] \right) \quad (13)$$

where we have stratified the map f by $X = \coprod_l S_l$, $0 \leq l \leq \dim X$, $\alpha_l : S_l \hookrightarrow X$ and the local system are given by $L_{i,l} = \alpha_l^* \mathcal{H}^{-l}(p^* \mathcal{H}^i(f_* \mathbb{Q}_X[n]))$. In this case, the stratification coincides with that induced by the ambient toric variety and the local systems $L_{i,l}$ are equal to those that occur in the decomposition theorem applied to $f : \mathbb{P}(\Delta_{\tilde{P}})_C \rightarrow \mathbb{P}(\Delta_P)$ and are there-

fore constant. Consequently, the cohomology sheaves of all terms in (13) give local families in a punctured disk around 0. Therefore, we may write down a monodromy operator and form the weight-monodromy filtration which is compatible with the isomorphism. The sheaf $\mathbb{Q}_{\tilde{X}}$ has the structure of a Hodge module, and hence by Saito's theory, the derived pushforwards $f_*\mathbb{Q}_{\tilde{X}}$ and $(p \circ f)_*\mathbb{Q}_{\tilde{X}}$ have the structure of mixed Hodge modules. Likewise, the relevant cohomology sheaves carry the structure of a mixed Hodge module [52], so their pushforwards do as well. See [35, Section 8.3.3] for an exposition. Consequently, the formula (13) is an isomorphism of mixed Hodge modules over C . Therefore, over a small punctured disk about 0, we get an isomorphism of admissible variations of mixed Hodge structures. \square

Note that the sum in the above theorem only needs to be over cones of Δ_P in the support of the recession fan of $\text{Trop}(X^\circ)$ because for other cones σ , X_σ° is empty and does not contribute.

6.2 Intersection cohomology of families of schön hypersurfaces of toric varieties

We will compute the refined limit Hodge–Deligne polynomial of schön, projective hypersurfaces using intersection cohomology. Our proof is inspired by that of [8] where one sums over strata in the stratification induced by the ambient toric variety and then constrains the intersection cohomology by Poincaré duality and the weak Lefschetz theorem.

Let $V \subset \mathbb{P}(\Delta_P)_\mathbb{C}$ be a closure of a schön hypersurface in an adapted projective toric variety defined by a lattice polytope P . We will need the following observations about the intersection cohomology of V :

- (a) The intersection cohomology of V obeys Poincaré duality [30].
- (b) The Hodge structure on $\text{IH}^*(V)$ is pure [52].
- (c) By the weak Lefschetz theorem, the Gysin map $\text{IH}^k(V) \rightarrow \text{IH}^{k+2}(\mathbb{P}(\Delta)_\mathbb{C})$ is a surjective map if $k \geq \dim V$ and is an isomorphism if $k > \dim V$. Moreover, it is a morphism of Hodge structures [30].

Now, we consider a family of closures of schön hypersurfaces X_t in $\mathbb{P} = \mathbb{P}(\Delta)_\mathbb{C}$ over a curve C that has a distinguished puncture such that $\mathbb{P}(\Delta_P)_\mathbb{C}$ is adapted for each X_t . By naturality, the monodromy around the puncture commutes with Poincaré duality and the Gysin map. We write $X = X_\mathbb{K}$ for X_t considered as a subvariety of $\mathbb{P}(\Delta_P)$ over \mathbb{K} .

We begin by writing down the refined limit Hodge–Deligne polynomials for $\mathbb{P}(\Delta_P)_\mathbb{K}$ and X . Because for each face Q of P , we have $\mathbb{P}(\Delta_P)_Q^\circ = (\mathbb{K}^*)^{\dim Q}$ and $E((\mathbb{P}(\Delta_P)_Q^\circ)_\infty; u, v, w) = (uvw^2 - 1)^{\dim Q}$, we have from Theorem 6.1:

$$E_{\text{int}}(\mathbb{P}_\infty; u, v, w) = \sum_{\substack{Q \subseteq P \\ Q \neq \emptyset}} (uvw^2 - 1)^{\dim Q} g([Q, P]^*; uvw^2)$$

$$E_{\text{int}}(X_\infty; u, v, w) = \sum_{\substack{Q \subseteq P \\ Q \neq \emptyset}} E((X_Q^\circ)_\infty; u, v, w) g([Q, P]^*; uvw^2).$$

Note that the first formula shows that $E_{\text{int}}(\mathbb{P}_\infty; u, v, w) = f(uvw^2)$, where f is the *toric h-polynomial* of P which is well known to give the dimensions of the topological intersection cohomology of the toric variety \mathbb{P} (see, e.g., [16, 26]).

We define $E_{\text{int,Lef}}$ by

$$E_{\text{int,Lef}}(X_\infty; u, v, w) = \sum_{m=0}^{\dim P-1} \sum_{p,q,r} (-1)^m h^{p,q,r} (IH^m(\mathbb{P}_\infty)) u^p v^q w^r \\ + \sum_{m=\dim P}^{2\dim P-2} \sum_{p,q,r} (-1)^m h^{p+1,q+1,r+2} (IH^{m+2}(\mathbb{P}_\infty)) u^p v^q w^r.$$

Note that this is a polynomial in uvw^2 . It represents the cohomology of X that we know must exist by the weak hyperplane theorem and Poincaré duality. Because all the relevant cohomology is of type (p, p) , the action of monodromy must be trivial by Remark 3.10. Then from the above expression for $E_{\text{int}}(\mathbb{P}_\infty; u, v, w)$ and Definition 4.3, one may deduce that

$$E_{\text{int,Lef}}(X_\infty; u, v, w) = E_{\text{int,Lef}}(P; uvw^2),$$

where $E_{\text{int,Lef}}(P; t)$ is defined by

$$(t-1)E_{\text{int,Lef}}(P; t) = t^{\dim P} g([\emptyset, P]^*; t^{-1}) - g([\emptyset, P]^*; t). \quad (14)$$

We define

$$E_{\text{int,prim}}(X_\infty; u, v, w) = E_{\text{int}}(X_\infty; u, v, w) - E_{\text{int,Lef}}(X_\infty; u, v, w).$$

This is the refined limit Hodge–Deligne polynomial corresponding to the primitive intersection cohomology in degree $\dim P - 1$,

$$IH_{\text{prim}}^{\dim P-1}(X_\infty) := \ker[IH^{\dim P-1}(X_\infty) \rightarrow IH^{\dim P+1}(\mathbb{P}_\infty)].$$

As in the proof of [8, Proposition 3.22], the induced Hodge structure (F, W) is pure and concentrated in W -degree $\dim P - 1$.

The following lemma establishes our main result (Corollary 1.8) on the intersection cohomology of families of schön, projective varieties. Explicitly, let $X^\circ \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, S) and $\dim P = n$. Let X denote the closure of X° in $\mathbb{P}(\Delta_P)_\mathbb{K}$. Then Theorem 1.5, which we proved in Sect. 5.4, states that

$$uvw^2 E(X_\infty^\circ; u, v, w) = (uvw^2 - 1)^{\dim P} + (-1)^{\dim P+1} h^*(P, S; u, v, w), \quad (15)$$

while Corollary 1.8 states that

$$uvw^2 E_{\text{int}}(X_\infty; u, v, w) = uvw^2 E_{\text{int,Lef}}(P; uvw^2) + (-1)^{\dim P+1} l^*(P, S; u, v) w^{\dim P+1}, \quad (16)$$

where $E_{\text{int,Lef}}(P; uvw^2)$ is given by (14).

Lemma 6.2 *Theorem 1.5 and Corollary 1.8 are equivalent.*

Proof Firstly, we show that (15) implies (16). Indeed, for every non-empty face Q of P , (15) implies

$$uvw^2 E((X_Q^\circ)_\infty; u, v, w) = (uvw^2 - 1)^{\dim Q} + (-1)^{\dim Q+1} h^*(Q, S|_Q; u, v, w).$$

We multiply this equation by $g([Q, P]^*; uvw^2)$ and sum over all such Q to obtain

$$E_{\text{int}}(X_{\infty}; u, v, w) = \sum_{\substack{Q \subseteq P \\ Q \neq \emptyset}} [(uvw^2 - 1)^{\dim Q} + (-1)^{\dim Q+1} h^*(Q, S|_Q; u, v, w)] \\ \times g([Q, P]^*; uvw^2),$$

where the left-hand side is computed using Theorem 6.1. Adding the equation $0 = -g([\emptyset, P]^*; uvw^2) + g([\emptyset, P]^*; uvw^2)$, and then simplifying using Definition 4.3 and Theorem 4.4, yields (16) as desired. We obtain (15) from (16) similarly. Explicitly, we claim that

$$E(X_{\infty}^{\circ}; u, v, w) = \sum_{\substack{Q \subseteq P \\ Q \neq \emptyset}} E_{\text{int}}((X_Q)_{\infty}; u, v, w) (-1)^{\dim P - \dim Q} g([Q, P]; uvw^2).$$

This follows by applying Theorem 6.1 to the right-hand side, together with Theorem 4.4. Now the above argument holds with $g([Q, P]^*; uvw^2)$ replaced by $(-1)^{\dim P - \dim Q} g([Q, P]; uvw^2)$. \square

We now give a new proof of (16) in the following equivalent form:

Corollary 6.3 *Let $\mathbb{K} = \mathbb{C}(t)$ and let $X^{\circ} \subseteq (\mathbb{K}^*)^n$ be a schön hypersurface, with associated Newton polytope and polyhedral subdivision (P, S) and $\dim P = n$. Let X denote the closure of X° in $\mathbb{P}(\Delta_P)_{\mathbb{K}}$. Then the refined limit Hodge–Deligne polynomial associated with the intersection cohomology of X is given by*

$$uvw^2 E_{\text{int,prim}}(X_{\infty}; u, v, w) = (-1)^{\dim P+1} w^{\dim P+1} l^*(P, S; u, v) \quad (17)$$

Equivalently,

$$l^*(P, S; u, v) = uv \sum_{p,q} h^{p,q} \left(IH_{\text{prim}}^{\dim P-1}(X_{\infty}) \right) u^p v^q.$$

Proof We have a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[u, v, w] & \xrightarrow[u \mapsto 1]{v \mapsto 1} & \mathbb{Z}[u, w] & & \\ \downarrow w \mapsto 1 & & \downarrow w \mapsto 1 & & \\ \mathbb{Z}[u, v] & \xrightarrow[v \mapsto 1]{} & \mathbb{Z}[u] & \xrightarrow[u \mapsto 1]{} & \mathbb{Z}. \end{array}$$

We will prove (17) by working our way through the diagram. In fact, we have proved the specialization of the equivalent statement (15) to $\mathbb{Z}[u]$ in Theorem 5.5.

In Sect. 5.4, we proved that if (15) holds when specialized to $\mathbb{Z}[u, w]$ then (15) holds when specialized to $\mathbb{Z}[u, v]$. Hence, we are left with the vertical arrows of the diagram. Because primitive cohomology is concentrated in W -degree equal to $\dim P - 1$, it is clear that if (17) holds for $\mathbb{Z}[u]$, then it holds for $\mathbb{Z}[u, w]$. Similarly, if (17) holds for $\mathbb{Z}[u, v]$, then it holds for $\mathbb{Z}[u, v, w]$. \square

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